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Maxwell–Bloch equations, C Neumann system and Kaluza–Klein theory

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Abstract

The Maxwell–Bloch equations are represented as the equation of motion for a continuous chain of coupled C Neumann oscillators on the three-dimensional sphere. This description enables us to find new Hamiltonian and Lagrangian structures of the Maxwell–Bloch equations. The symplectic structure contains a topologically non-trivial magnetic term which is responsible for the coupling. The coupling forces are geometrized by means of an analogue of Kaluza–Klein theory. The conjugate momentum of the additional degree of freedom is precisely the speed of light in the medium. It can also be thought of as the strength of the coupling. The Lagrangian description has a structure similar to that of the Wess–Zumino–Witten–Novikov action. We describe two families of solutions of the Maxwell–Bloch equations which are expressed in terms of the C Neumann system. One family describes travelling non-linear waves whose constituent oscillators are the C Neumann oscillators in the same way as the harmonic oscillators are the constituent oscillators of the harmonic waves. The 2π-pulse soliton is a member of this family.

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1. Introduction

The Maxwell–Bloch equations are a well-known system of partial differential equations used in non-linear optics. Roughly speaking, these equations are a semi-classical model of the resonant interaction between light and an active optical medium consisting of two-level atoms. We will consider the following form of the Maxwell–Bloch equations without pumping or broadening:

$$E_t + cE_x = P - \alpha E, \qquad P_t = ED - \beta P, \qquad D_t = -\tfrac{1}{2}(\overline{E}P + E\overline{P}) - \gamma(D-1). \quad (1)$$

The independent variables x and t parametrize one spatial dimension and the time, the complex-valued functions $E(t,x)$ and $P(t,x)$ describe the slowly varying envelopes of the electric field and the polarization of the medium, respectively, and the real-valued function D describes the level inversion. The constant c is the speed of light in the medium, α represents the losses of the electric field, while β is the longitudinal and γ the transverse relaxation rate in the medium. In our paper we shall assume that $\alpha = \gamma = 0$. We shall consider the spatially periodic case of (1). The Maxwell–Bloch equations are an integrable system (see [1–4]). In particular, they satisfy the zero curvature condition.

The other integrable system which figures in this paper is the C Neumann system. The C Neumann system describes the motion of a particle on the n-dimensional sphere S^n under the influence of the force whose potential is quadratic. This oscillator was first described in the 19th century by Carl Neumann in [5]. More recently, many authors studied its different geometrical aspects. See [6–8] and many other texts. We will show that there is an interesting relationship between the Maxwell–Bloch equations and the C Neumann oscillator. Results of this paper are motivated by this relationship.

The Hamiltonian system $(T^*SU(2), \omega_c, H_{cn})$, where the function $H_{cn}: T^*SU(2) \to \mathbb{R}$ is given by

$$H_{cn}(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + \mathrm{Tr}(\sigma \cdot \mathrm{Ad}_q(\tau)), \qquad \sigma, \tau \in \mathfrak{su}(2), \tag{2}$$

describes the C Neumann oscillator moving on the 3-sphere $S^3 = SU(2)$. The force potential is given by a quadratic form on $\mathbb{R}^4$ whose 4×4 symmetric matrix has two double eigenvalues. Our theorem 1 claims that the Maxwell–Bloch equations describe a continuous chain of interacting C Neumann oscillators of the above type. The oscillators in the chain are parametrized by the spatial dimension of the Maxwell–Bloch equations and the interaction between the oscillators is of *magnetic type*. By this we mean that the acceleration of a given oscillator is influenced by the velocity and not by the position of the neighbouring oscillators. More concretely, the Maxwell–Bloch equations (1) are the equations of motion for the Hamiltonian system $(T^*LSU(2), \omega_c + c\omega_m, H_{mb})$. Here $LSU(2) = \{g: S^1 \to SU(2)\}$ is the loop group of $SU(2)$ and the Hamiltonian function $H_{mb}: T^*LSU(2) \to \mathbb{R}$ is given by

$$H_{mb}(g, p_g) = \int_{S^1} \left(\frac{1}{2}\|p_g(x)\|^2 + \mathrm{Tr}(\sigma \cdot \mathrm{Ad}_{g(x)}(\tau(x)))\right) \mathrm{d}x.$$

We see that the Hamiltonian is precisely the total energy of our chain of the C Neumann oscillators parametrized by $x \in S^1$. The symplectic form $\omega_c + c\omega_m$ is a perturbation of the canonical form ω_c. The perturbation term ω_m is the natural pull-back of the 2-form Ω_m on $LSU(2)$ which is right invariant on $LSU(2)$ and whose value at the identity $e \in LSU(2)$ is given by

$$\Omega_m(\xi, \eta) = \int_{S^1} \mathrm{Tr}(\xi'(x) \cdot \eta(x))\, \mathrm{d}x, \qquad \xi(x), \eta(x) \in L\mathfrak{su}(2) = T_e LSU(2).$$

The term ω_m is responsible for the magnetic-type interaction among the neighbouring oscillators in our chain.

At the level of the equations of motion, the relationship between the C Neumann system and the Maxwell–Bloch equations is reflected in the following. The equation of motion of the C Neumann system $(T^*SU(2), \omega_c, H_{cn})$ is

$$(g_t g^{-1})_t = [\sigma, \mathrm{Ad}_g(\tau)]; \qquad g(t) : I \longrightarrow SU(2),$$

while the Maxwell–Bloch equations can be rewritten in the form

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_(\tau(x))]; \qquad g(t,x): I \longrightarrow SU(2). \tag{3}$$

More precisely, the above equation is equivalent to the system (1), if we impose the constraint $\mathrm{Tr}(g_t g^{-1} \cdot \sigma) = \mathrm{const}$. The rewriting (3) shows clearly that the stationary (time-independent) solutions of the Maxwell–Bloch equations are solutions of our C Neumann equation. In this paper, we consider equation (3) without the constraint. This makes the discussion easier and clearer. In addition, we believe that equation (3), being a description of a chain of oscillators, is interesting in itself.

A more interesting illustration of the relationship between the Maxwell–Bloch equations and the C Neumann system is provided by the solutions of the former given in proposition 3. These solutions are the non-linear travelling waves whose constituent oscillator is the C Neumann oscillator in the same way as the harmonic oscillator is the constituent oscillator of the harmonic waves. More precisely, the constituent oscillator turns out to be the electrically charged spherical pendulum moving in the field of the magnetic monopole which is positioned at the centre of the sphere. For small oscillations of the spherical pendulum, our solutions indeed behave similarly as the harmonic waves. (Indeed, the linearization around the stable equilibrium of our equation yields the harmonic waves.) But we show in section 4 that the famous 2π-pulse soliton is a particular case of the solutions given in proposition 3. This solution occurs when the constituent oscillator becomes the planar gravitational pendulum. In addition, its energy must be the energy of the separatrix of the pendulum's phase portrait.

The difference between the symplectic structure of a Hamiltonian system and the canonical structure is called the magnetic term. The momentum shifting argument (see e.g. [9] or [10]) tells us that the magnetic term is responsible for a force which depends linearly on the momenta. An example is the Lorentz force of a magnetic field acting on a moving charged particle. Geometrization of such forces can be achieved by analogues of the Kaluza–Klein theory. This approach provides the configuration space in which the motion of a charged particle under the influence of the magnetic force is described by the geodesic motion. In Hamiltonian terms, this means that the relevant symplectic structure will be canonical. The geometrization is achieved by the introduction of an additional circular degree of freedom. The extended configuration space is thus a $U(1)$-bundle over the original configuration space. A key role is played by the connection which is given on this bundle and whose curvature is precisely the magnetic term. In symplectic geometry, the procedure of adding degrees of freedom and their conjugate momenta is called the symplectic reconstruction—a process inverse to the symplectic reduction. Symplectic reconstruction was studied e.g. in [9, 11, 12]. In the case of the Lorentz force, the moment conjugate to the (single) additional dimension is precisely the electric charge of the moving particle. Therefore, the new momentum is usually called the charge. We shall see that in the case of the Maxwell–Bloch equations the role of the Kaluza–Klein charge is taken by the speed of light.

In our case, the magnetic term ω_{m} is not exact. The class $[\Omega_{\mathrm{m}}]$ is a non-zero element in the cohomology group $H^2(LSU(2))$. In such cases, the idea of the Kaluza–Klein geometrization has to be used with some care. It can be performed only when the magnetic term is an integral 2-form. This follows from a well-known theorem of A Weil. Our proposition 5 claims that, in general, whenever the magnetic term σ_{m} of a system $(T^*N, \omega_{\mathrm{c}} + \sigma_{\mathrm{m}}, H)$ is integral, there exists the extended Hamiltonian system $(T^*M, \Omega_{\mathrm{c}}, \widetilde{H})$ whose configuration space is the total space of a $U(1)$-bundle $M \to N$. The extended system is invariant with respect to the natural $U(1)$-action and $(T^*N, \omega_{\mathrm{c}} + \sigma_{\mathrm{m}}, H)$ is its symplectic quotient. The class $[\sigma_{\mathrm{m}}] \in H^2_{\mathrm{DR}}(N)$ is the Chern class of $M \to N$. Our theorem 2 describes the Kaluza–Klein description of the Maxwell–Bloch system. Let $\widetilde{LSU}(2)$ be the central extension of the loop group $LSU(2)$. Let the Hamiltonian function $\widetilde{H}$ of the system $(T^*\widetilde{LSU}(2), \Omega_{\mathrm{c}}, \widetilde{H}_{\mathrm{mb}})$ be

$$\widetilde{H}_{\mathrm{mb}}(\widetilde{g}, p_{\widetilde{g}}) = \frac{1}{2}\|p_{\widetilde{g}}\|^2 + \int_{S^1} \mathrm{Tr}\Big(\sigma \cdot \mathrm{Ad}_{\widetilde{g}}(\tau(x))\Big)\,\mathrm{d}x,$$

where $\|p_{\widetilde{g}}\|$ is given by the natural metric on the central extension $\widetilde{L\mathfrak{su}}(2) = L\mathfrak{su}(2) \oplus \mathbb{R}$. Then this system is invariant with respect to the natural $U(1)$-action. Its symplectic quotient at the level c of the momentum map is the Maxwell–Bloch Hamiltonian system $(T^*LSU(2), \omega_c + c\omega_m, H_{mb})$ on $LSU(2)$. We note that $S^1 \to \widetilde{LSU}(2) \to LSU(2)$ is a non-trivial $U(1)$-bundle whose first Chern class is $[\Omega_m] \in H^2(LSU(2))$. The charge in the Kaluza–Klein description $(T^*\widetilde{LSU}(2), \Omega_c, \widetilde{H}_{mb})$ of the Maxwell–Bloch system has a clear physical interpretation. It is precisely the *speed of light* in the medium in question. Alternatively, it can be thought of as the strength of the coupling among the neighbouring C Neumann oscillators.

The situation described above is reminiscent of the following finite-dimensional one. Let $(T^*SU(2), \Omega_c, H_{cn})$ be the C Neumann system on $SU(2) = S^3$, with the Hamiltonian given by (2). This system is invariant with respect to the $U(1)$-action which arises from the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ given by $g \to \mathrm{Ad}_g(\tau)$. The symplectic quotient is $(T^*S^2, \omega_c + \omega_m, H_{sp})$, where

$$H_{sp}(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + \mathrm{Tr}(\sigma \cdot q)$$

and ω_m is the pull-back of the volume form Ω_m on S^2. This system describes the spherical pendulum in the magnetic field of the Dirac monopole situated at the centre of S^2. The form $[\Omega_m] \in H^2(S^2)$ is the first Chern class of the Hopf fibration. This construction is described in more detail in [13] and in greater generality in [14].

An important merit of the Kaluza–Klein approach lies in the fact that it clarifies the otherwise elusive Lagrangian description of the systems with non-trivial magnetic terms. In theorem 3, we give the Lagrangian expression of the Maxwell–Bloch system on the extended configuration space $\widetilde{LSU}(2)$. The proof is a straightforward application of the Legendre transformation. We stress the fact that the Lagrangian description of a solution, which is not periodic in time, is possible *only* on the extended configuration space. The presence of the topologically non-trivial magnetic term makes the Lagrangian description on the primary configuration space $LSU(2)$ more involved. This description is given in theorem 4. The Lagrangian has a structure similar to that of the Wess–Zumino–Witten–Novikov Lagrangian. In particular, it is well defined only for those solutions of the Maxwell–Bloch equations which are temporally periodic. We note that the results and proofs of section 5 hold with only minor notational changes for a general Hamiltonian system with a non-trivial (but integral) magnetic term.

Throughout this paper, the group $SU(2)$ can be replaced by any compact semi-simple Lie group G. Thus, our construction yields a family of integrable infinite-dimensional systems $(T^*LG, \omega_c + c\omega_m, H_{gmb})$ which satisfy the zero-curvature condition. All these integrable systems are systems with the non-trivial magnetic term $\omega_m \in \Omega^2(LG)$ and with the geometric phase.

The rewriting (3) of the Maxwell–Bloch equations is already used in the papers [15, 16][1] by Park and Shin. There it is interpreted as an equation of a field theory. The connection between the principal chiral field theories on the one hand and the Maxwell–Bloch equations, or more precisely, the self-induced transparency theory of McCall and Hahn, on the other hand, was already established by Maimistov in [17]. The authors of [15, 16] find the Lagrangian of the Maxwell–Bloch equations by means of field-theoretic considerations. Our WZWN-type Lagrangian from theorem 4 is essentially the same as the one found by Park and Shin. The only difference is that we consider the unconstrained equation (3), while Park and Shin take the constraint $\mathrm{Tr}(g_t g^{-1} \cdot \sigma) = \mathrm{const}$ into account. They very elegantly and ingeniously

[1] References [15, 16] were brought to the author's attention by the referees after the submission of this paper. The author was previously not aware of the existence of these two important papers.

subsume this constraint into the $U(1)$-gauging part of the WZWN theory. The rewriting (3) enables Park and Shin to describe many important features of the Maxwell–Bloch equations, including soliton numbers, conserved topological and non-topological charges, as well as certain symmetry issues. In [16], they also show that the above-mentioned generalizations of equation (3) to Lie groups G other than $SU(2)$ are, in some cases, relevant to the theory of the resonant light–matter interaction. In particular, they show explicitly that various non-degenerate and degenerate two- and three-level light–matter systems can be described by equation (3) with the appropriate choice of the group G and of the constant τ. Certain choices of these two constants give rise to the systems whose configuration spaces are supported on symmetric spaces of the form G/H, where $H \subset G$ is a suitable subgroup. In terms of our Hamiltonian description, these systems are precisely the symplectic quotients of $(T^*LG, \omega_c + c\omega_m, H_{gmb})$ with respect to the natural action of LH.

2. A rewriting of the Maxwell–Bloch system

In this section we shall express the Maxwell–Bloch equations in a form which will reveal their connection with the C Neumann system.

Let the functions $E(t,x)$ and $P(t,x)$ be complex valued and let the values of $D(t,x)$ be real. We shall consider the Maxwell–Bloch equations

$$E_t + cE_x = P, \qquad P_t = ED - \beta P, \qquad D_t = -\tfrac{1}{2}(\overline{E}P + E\overline{P}), \tag{4}$$

with spatially periodic boundary conditions:

$$E(t, x+2\pi) = E(t,x), \qquad P(t,x+2\pi) = P(t,x), \qquad D(t,x+2\pi) = D(t,x). \tag{5}$$

The system (4) can be rewritten in a more compact form. Let the Lie algebra-valued maps $\rho(t,x)\colon \mathbb{R}\times S^1 \to \mathfrak{su}(2)$ and $F(t,x)\colon \mathbb{R}\times S^1 \to \mathfrak{su}(2)$ be defined as

$$\rho(t,x) = \begin{pmatrix} \mathrm{i}D(t,x) & \mathrm{i}P(t,x) \\ -\mathrm{i}\overline{P}(t,x) & -\mathrm{i}D(t,x) \end{pmatrix}, \qquad F(t,x) = \frac{1}{2}\begin{pmatrix} \mathrm{i}\beta & E(t,x) \\ -\overline{E}(t,x) & -\mathrm{i}\beta \end{pmatrix}. \tag{6}$$

In terms of these maps, the system (4) acquires the form

$$\rho_t = [\rho, F], \qquad F_t + cF_x = [\rho, \sigma], \tag{7}$$

where

$$\sigma = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

We observe that the equation $\rho_t = [\rho, F]$ is of the Lax form. Therefore, we have

$$\rho(t,x) = \mathrm{Ad}_{g(t,x)}(\tau(x)), \qquad F(t,x) = -g_t(t,x)\cdot g^{-1}(t,x), \tag{8}$$

where $\tau(x)\colon S^1 \to \mathfrak{su}(2)$ and $g(t,x)\colon \mathbb{R}\times S^1 \to SU(2)$ are arbitrary smooth matrix-valued functions. If we insert the above into the second equation of the system (7), we obtain the following second-order partial differential equation for $g(t,x)\colon \mathbb{R}\times S^1 \to SU(2)$:

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))]. \tag{9}$$

This is the new rewriting of the Maxwell–Bloch equations that we shall use in this paper. Equation (9) is slightly more general than the Maxwell–Bloch equations (4). It is equivalent to (4), if we add the stipulation

$$\langle g_t g^{-1}, \sigma\rangle = \mathrm{const} = -\beta.$$

We will consider equation (9) as an equation of motion for the group-valued loop $g(x)(t) = g(t,x) \in \{S^1 \to SU(2)\} = LSU(2)$, where $LSU(2)$ denotes the loop group of unbased $SU(2)$ loops. In other words, a solution of equation (9) is a path

$$g(t,x)\colon I \longrightarrow LSU(2), \qquad t \longmapsto g(t,x).$$

Then for every choice of the loop $\tau(x)\colon S^1 \to \mathfrak{su}(2)$, together with a choice of the initial conditions $g(0,x) \in LSU(2)$ and $g_t(0,x)\cdot g^{-1}(0,x) \in L\mathfrak{su}(2)$, we expect solutions $g(t,x)$ of (9). By $L\mathfrak{su}(2)$ we denoted the loop algebra $L\mathfrak{su}(2) = \{\tau\colon S^1 \to \mathfrak{su}(2)\}$ which is, of course, the Lie algebra of $LSU(2)$.

We conclude this section by pointing out that our rewriting of the Maxwell–Bloch equation yields a whole family of integrable partial differential equations. Let G be an arbitrary semi-simple Lie group and let $g(t,x)\colon I \times S^1 \to G$ be a smooth map. Let us put $c = 1$. A straightforward check gives the proof of the following proposition.

Proposition 1. *Let $\sigma \in \mathfrak{g}$ be an arbitrary element and let $\tau\colon S^1 \to \mathfrak{g}$ be a loop in the Lie algebra $\mathfrak{g}$. The equation*

$$(g_t g^{-1})_t + (g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))]$$

satisfies the zero-curvature condition:

$$V_t - U_x + [U,V] = 0,$$

where

$$U = -(-z\sigma + g_t g^{-1}) \qquad \text{and} \qquad V = -z\sigma + g_t g^{-1} - \frac{1}{z}\mathrm{Ad}_g(\tau).$$

3. Hamiltonian structure with the magnetic term

We shall now take a closer look at the equation

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))].$$

Consider first those solutions $g(t)\colon I \to SU(2)$ of (9) which are constant with respect to the x-variable. Clearly, such solutions will exist only in the case when $\tau(x) \equiv \tau$ is a constant element in $\mathfrak{su}(2)$. The Lie group-valued function $g(t)$ will then be a solution of the ordinary differential equation

$$g_t g^{-1} = [\sigma, \mathrm{Ad}_g(\tau)]. \tag{10}$$

For $\alpha, \beta \in \mathfrak{su}(2)$, let $\langle \alpha, \beta\rangle = -\frac{1}{2}\mathrm{Tr}(\alpha\cdot\beta)$ denote the Killing form on $\mathfrak{su}(2)$.

Proposition 2. *Equation (10) is the equation of motion for the Hamiltonian system $(T^*SU(2), \omega_c, H_{cn})$, where the Hamiltonian is given by*

$$H_{cn}(g, p_g) = \tfrac{1}{2}\|p_g\|^2 + \langle \sigma, \mathrm{Ad}_g(\tau)\rangle \tag{11}$$

and ω_c is the canonical symplectic form on the cotangent bundle $T^*SU(2) = T^*S^3$.

This system is a special case of the C Neumann oscillator on the 3-sphere. In the suitably chosen Cartesian co-ordinates on $\mathbb{R}^4$, the potential of H_{cn} assumes the form

$$\langle \sigma, \mathrm{Ad}_{g(\vec{q})}(\tau)\rangle = \lambda\big(q_1^2 + q_2^2\big) - \lambda\big(q_3^2 + q_4^2\big),$$

where λ is a positive real number.

Proof. First we shall prove that H_{cn} is indeed the Hamiltonian of equation (10) with respect to the canonical symplectic form. Let G be an arbitrary compact semi-simple Lie group and

T^*G its cotangent bundle. Let $T^*G \cong G \times \mathfrak{g}^*$ be the trivialization by means of the right translations. In this trivialization, the canonical simplectic form ω_c on T^*G is given by the formula

$$(\omega_c)_{(g,p_g)}((X_b, X_{ct}), (Y_b, Y_{ct})) = -\langle X_{ct}, Y_b\rangle + \langle Y_{ct}, X_b\rangle + \langle p_g, [X_b, Y_b]\rangle. \tag{12}$$

Above, $\langle a, x\rangle$ denotes the evaluation of the element $a \in \mathfrak{g}^*$ on the element $x \in \mathfrak{g}$. For the proof see [18].

Let (M, ω, H) be a Hamiltonian system on the symplectic manifold (M, ω). A path $\gamma(t)\colon I \to M$ is a solution of the equation of motion for our system, if $\dot\gamma(t) = X_H(\gamma(t))$, where X_H is the Hamiltonian vector field defined by $\mathrm{d}H = \omega(X_H, -)$.

For the Hamiltonian given by (11), we have

$$\langle \mathrm{d}H_{cn}, (\delta_b, \delta_{ct})\rangle = -\langle [\sigma, \mathrm{Ad}_g(\tau)]^\flat, \delta_b\rangle + \big\langle \delta_{ct}, p_g^\sharp\big\rangle. \tag{13}$$

Here $\flat\colon \mathfrak{g} \to \mathfrak{g}^*$ and $\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ are defined by $\alpha^\flat = \langle \alpha, -\rangle$ and $\beta = \langle \beta^\sharp, -\rangle$. Let us denote $X_{H_{cn}} = (X_b, X_{ct}) \in \Gamma(T^*SU(2)) = \Gamma(SU(2) \times \mathfrak{su}(2)^*)$, where we use the trivialization by the right translations. Then

$$\begin{aligned}(\omega_c)_{(g,p_g)}((X_b, X_{ct}), (\delta_b, \delta_{ct})) &= -\langle X_{ct}, \delta_b\rangle + \langle \delta_{ct}, X_b\rangle + \langle p_g, [X_b, \delta_b]\rangle \\ &= \langle -X_{ct} - \{X_b, p_g\}, \delta_b\rangle + \langle \delta_{ct}, X_b\rangle\end{aligned} \tag{14}$$

and $\{a, \alpha\}$ denotes the ad^*-action of $a \in \mathfrak{su}(2)$ on $\alpha \in \mathfrak{su}(2)^*$. Comparing (13) and (14), we obtain

$$p_g^\sharp = X_b, \qquad [\sigma, \mathrm{Ad}_g(\tau)]^\flat = X_{ct} + \{X_b, p_g\}$$

and from this

$$X_b = p_g^\sharp, \qquad X_{ct} = [\sigma, \mathrm{Ad}_g(\tau)]^\flat.$$

Let $\gamma(t) = (g(t), p_g(t))\colon I \to T^*G$ be a path and let $\dot\gamma = (g_t g^{-1}, (p_g)_t)$ be its tangent at (g, p_g) expressed in the right trivialization. Then the above equations and $(g_t g^{-1}, (p_g)_t) = (X_b, X_{ct})$ give us

$$(g_t g^{-1})_t = [\sigma, \mathrm{Ad}_g(\tau)],$$

which proves the first part of our proposition.

The proof of the second part is a matter of simple checking. An element $g \in SU(2)$ is a matrix of the form

$$g = \begin{pmatrix} g_1 + \mathrm{i}g_2 & g_3 + \mathrm{i}g_4 \\ -g_3 + \mathrm{i}g_4 & g_1 - \mathrm{i}g_2\end{pmatrix}, \qquad \det(g) = \sum_{i=1}^4 g_i^2 = 1.$$

Let

$$\tau = \begin{pmatrix} \mathrm{i}a & b + \mathrm{i}c \\ -b + \mathrm{i}c & -\mathrm{i}a\end{pmatrix}.$$

Then, $\langle \sigma, \mathrm{Ad}_g(\tau)\rangle$ is the quadratic form

$$\begin{aligned}\langle \sigma, \mathrm{Ad}_g(\tau)\rangle &= -\mathrm{Tr}(\sigma g \tau g^{-1}) \\ &= 2a\big(g_1^2 + g_2^2 - g_3^2 - g_4^2\big) + 4b(-g_1g_4 + g_2g_3) + 4c(g_1g_3 + g_2g_4)\end{aligned}$$

on $\mathbb{R}^4$ restricted to the sphere $SU(2) = S^3 \subset \mathbb{R}^4$. The 4×4-matrix of this quadratic form has two double eigenvalues

$$\lambda = 2\sqrt{a^2 + b^2 + c^2} \qquad \text{and} \qquad \mu = -\lambda = -2\sqrt{a^2 + b^2 + c^2},$$

which concludes the proof of the proposition. □

Let us now return to equation (9):

$$(g_t g^{-1})_t = -c(g_t g^{-1})_x + [\sigma, \mathrm{Ad}_g(\tau(x))].$$

This can now be thought of as the equation of motion of a continuous chain of C Neumann oscillators parametrized by $x \in S^1$. At the time t, the position of the x_0th oscillator is $g(t, x_0) \in SU(2) \cong S^3$. The above equation can be written in the form

$$(g_t g^{-1})_t(x) = -\frac{c}{2\epsilon}(g_t g^{-1}(x-\epsilon) - g_t g^{-1}(x+\epsilon))|_{\epsilon\to 0} + [\sigma, \mathrm{Ad}_{g(x)}(\tau(x)].$$

For every x, the acceleration of the oscillator $g(t, x)$ is determined by the potential $[\sigma, \mathrm{Ad}_{g(x)}(\tau(x))]$ and by the velocities $g_t g^{-1}(x \pm \epsilon)$ of the infinitesimally close oscillators. The interaction of the neighbouring oscillators is of magnetic type. It depends on the velocities of the particles and not on their positions.

This interpretation of the Maxwell–Bloch equation suggests a Hamiltonian structure. The configuration space is the space of positions of the continuous C Neumann chains. This is the space of maps $g(x)\colon S^1 \to SU(2)$, in other words, the loop group $LSU(2)$. Thus, the phase space will be the cotangent bundle $T^*LSU(2)$. The natural choice of the Hamiltonian is the total energy of all the oscillators:

$$H_{\mathrm{mb}}(g(x), p_g(x)) = \int_{S^1}\left(\frac{1}{2}\|p_g(x)\|^2 + \langle\sigma, \mathrm{Ad}_{g(x)}(\tau(x))\rangle\right)\mathrm{d}x. \tag{15}$$

Let ω_{c} now denote the canonical cotangent symplectic structure on $T^*LSU(2)$. It is easily seen that the equation of motion of the Hamiltonian system $(T^*LSU(2), \omega_{\mathrm{c}}, H_{\mathrm{cn}})$ is simply $(g_t g^{-1})_t = [\sigma, \mathrm{Ad}_g(\tau(x))]$. Therefore, the canonical symplectic form ω_{c} has to be perturbed by a form which will account for the interaction term $(g_t g^{-1})_x$.

Let $(\Omega_{\mathrm{m}})_e$ be the skew bilinear form on the loop algebra $L\mathfrak{su}(2)$ given by the formula

$$(\Omega_{\mathrm{m}})_e(\xi, \eta) = \int_{S^1}\langle\eta_x, \xi\rangle\,\mathrm{d}x = -\int_{S^1}\langle\xi_x, \eta\rangle\,\mathrm{d}x, \qquad \xi(x), \eta(x) \in L\mathfrak{su}(2).$$

This bilinear form is a Lie algebra cocycle. Let Ω_{m} be the right-invariant 2-form on $LSU(2)$ whose value at the identity $e \in LSU(2)$ is $(\Omega_{\mathrm{m}})_e$. Since $(\Omega_{\mathrm{m}})_e$ is a cocycle, the form Ω_{m} is closed. Let $proj\colon T^*LSU(2) \to LSU(2)$ be the natural projection and denote the pull-back $proj^*(\Omega_{\mathrm{m}})$ by ω_{m}. The form ω_{m} is then a closed differential 2-form on $T^*LSU(2)$.

Theorem 1. *Let $(T^*LSU(2), \omega_{\mathrm{c}} + c\omega_{\mathrm{m}}, H_{\mathrm{mb}})$ be the Hamiltonian system, where the Hamiltonian H is given by (15), the form ω_{c} is the canonical cotangent form and ω_{m} is the form described above. Then the equation of motion is the Maxwell–Bloch equation*

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))].$$

Proof. Let $\xi(x)$ and $\eta(x)$ be two arbitrary elements of the loop Lie algebra $L\mathfrak{su}(2)$. The inner product on $L\mathfrak{su}(2)$ defined by the formula

$$\langle\langle\xi(x), \eta(x)\rangle\rangle = \int_{S^1}\langle\xi(x), \eta(x)\rangle\,\mathrm{d}x$$

is nondegenerate and Ad-invariant with respect to the group $LSU(2)$. By the same symbol we shall also denote the evaluation $\langle\langle\alpha, a\rangle\rangle$ of the element $\alpha \in L\mathfrak{su}(2)^*$ on an element $a \in L\mathfrak{su}(2)$, as well as the induced inner product on $L\mathfrak{su}(2)^*$. Thus, the Hamiltonian (15) can be written in the form

$$H_{\mathrm{mb}}(g, p_g) = \tfrac{1}{2}\|p_g\|^2 + \langle\langle\sigma, \mathrm{Ad}_g(\tau)\rangle\rangle,$$

where $\|p_g\|^2 = \langle\langle p_g, p_g\rangle\rangle$. The canonical cotangent form on $T^*LSU(2)$ has the expression analogous to (12), namely

$$(\omega_c)_{(g,p_g)}((X_b, X_{ct}), (Y_b, Y_{ct})) = -\langle\langle X_{ct}, Y_b\rangle\rangle + \langle\langle Y_{ct}, X_b\rangle\rangle + \langle\langle p_g, [X_b, Y_b]\rangle\rangle, \tag{16}$$

where $(X_b, X_{ct}), (Y_b, Y_{ct})$ is an arbitrary pair of tangent vectors from $T_{(g,p_g)}(T^*LSU(2))$ written in the right trivialization. The expression of the symplectic form $\omega_c + c\omega_m$ in this trivialization is

$$\begin{aligned}(\omega_c + c\omega_m)_{(g,p_g)}((X_b, X_{ct}), (Y_b, Y_{ct})) &= -\langle\langle X_{ct}, Y_b\rangle\rangle + \langle\langle Y_{ct}, X_b\rangle\rangle \\ &\quad + \langle\langle p_g, [X_b, Y_b]\rangle\rangle - c\langle\langle (X_b)_x, Y_b\rangle\rangle.\end{aligned}$$

Similarly as in the proof of proposition 2, we have

$$\langle\langle \mathrm{d}H_{mb}, (\delta_b, \delta_{ct})\rangle\rangle = -\langle\langle [\sigma, \mathrm{Ad}_g(\tau)]^\flat, \delta_b\rangle\rangle + \langle\langle \delta_{ct}, p_g^\sharp\rangle\rangle$$

and

$$\begin{aligned}&(\omega_c + c\omega_m)_{(g,p_g)}((X_b, X_{ct}), (\delta_b, \delta_{ct})) \\ &\quad = -\langle\langle X_{ct}, \delta_b\rangle\rangle + \langle\langle \delta_{ct}, X_b\rangle\rangle + \langle\langle p_g, [X_b, \delta_b]\rangle\rangle - c\langle\langle (X_b)_x, \delta_b\rangle\rangle \\ &\quad = \left\langle\!\left\langle -X_{ct} - c(X_b)_x^\flat - \{X_b, p_g\}, \delta_b\right\rangle\!\right\rangle + \langle\langle \delta_{ct}, X_b\rangle\rangle.\end{aligned}$$

Again, because of the independence of δ_b and δ_{ct}, the above two equations give

$$p_g^\sharp = X_b, \qquad X_{ct} + c(X_b)_x^\flat = [\sigma, \mathrm{Ad}_g(\tau)]^\flat. \tag{17}$$

Solutions of the Hamiltonian system $(T^*LSU(2), \omega_c + c\omega_m, H_{mb})$ are the paths

$$\gamma(t;x) = (g(t;x), p_g(t;x)) : I \longrightarrow T^*LSU(2) \cong LSU(2) \times (L\mathfrak{su}(2))^*,$$

which are the integral curves of the Hamiltonian vector field $X_{H_{mb}}$ of the Hamiltonian H. The condition $(g_t g^{-1}, (p_g)_t) = (X_b, X_{ct})$ and equations (17) finally give

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau)],$$

which proves our theorem. □

It is clear that the above theorem holds if the group $SU(2)$ is replaced by any compact semi-simple Lie group G. Every such G is endowed with the Killing form $\langle -, -\rangle$ and the cocycle

$$\omega_m(\xi, \eta) = -\int_{S^1} \langle \xi_x, \eta\rangle, \qquad \xi, \eta \in L\mathfrak{g},$$

on the corresponding loop algebra is well defined. The equation

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x)]$$

for $g(t, x) : I \times S^1 \to G$ is the equation of motion of the system $(T^*LG, \omega_c + c\omega_m, H_{gmb})$, where H_{gmb} and ω_m are defined in the same way as above. (By H_{gmb} we denoted the Hamiltonian of the generalized Maxwell–Bloch system.) This system describes a continuous chain of oscillators on G given by (T^*G, ω_c, H_{rs}), where

$$H_{rs}(g, p_g) = \tfrac{1}{2}\|p_g\|^2 + \langle\sigma, \mathrm{Ad}_g(\tau)\rangle.$$

These are the well-known integrable systems described by Reymann and Semenov-Tian-Shansky in [19, 20]. Connection of such systems with Nahm's equations of the Yang–Mills theory is studied in [21].

4. Two families of solutions

In this section we omit the spatial periodicity condition. It will be convenient to work with the symplectic reduction of our C Neumann system which was already mentioned in the introduction.

Let us denote the position variable of the C Neumann system $(T^*SU(2), \omega_c, H_{cn})$ by $h \in SU(2)$. The corresponding equation of motion is

$$(h_t h^{-1})_t = [\sigma, \mathrm{Ad}_h(\tau)]. \tag{18}$$

This system is invariant with respect to the actions of the circle groups $U_\tau(1) = \{\mathrm{Exp}(s \cdot \tau)\}$ and $U_\sigma(1) = \{\mathrm{Exp}(s \cdot \sigma)\}$ in $SU(2)$. The action of $U_\tau(1)$ is the cotangent lift of the action $(\rho_\tau)_u(h) = h \cdot u$ on $SU(2)$. In [13], we show that the moment map $\mu: T^*SU(2) \to \mathfrak{u}(1)^*$ is given by

$$\mu(h, h_t h^{-1}) = \langle h_t h^{-1}, \mathrm{Ad}_h(\tau)\rangle. \tag{19}$$

Above, we identified the cotangents and tangents by means of the Riemannian metric and we shall continue to do so below. The symplectic quotient of $(T^*SU(2), \omega_c, H_{cn})$ with respect to ρ_τ at the level m of the moment map μ is the Hamiltonian system $(T^*S^2_\tau, \omega_c + m\omega_{dm}, H_{sp})$, where

$$H_{sp}(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + \langle q, \sigma\rangle.$$

Here $q = \mathrm{Ad}_h(\tau) \in S^2_\tau \subset \mathfrak{su}(2) = \mathbb{R}^3$. This system describes the charged spherical pendulum moving on the 2-sphere S^2_τ under the influence of the gravitational force potential $\langle \sigma, q\rangle$ and the Lorentz force caused by the Dirac magnetic monopole positioned at the centre of S^2_τ. The charge of the pendulum is m. This system is described in more detail in [13].

The differentiation $q_t = [h_t h^{-1}, \mathrm{Ad}_h(\tau)] = [h_t h^{-1}, q]$ and the fact that the map

$$[-, q] : T_q S^2_\tau \longrightarrow T_q S^2_\tau; \qquad v \longmapsto [v, q]$$

is a rotation through $\frac{\pi}{2}$, give us the expression

$$h_t h^{-1} = -[q_t, q] + \langle h_t h^{-1}, q\rangle q = -[q_t, q] + mq. \tag{20}$$

Since $\langle h_t h^{-1}, \sigma\rangle_t = \langle (h_t h^{-1})_t, \sigma\rangle$, it is now clear from (18) that

$$\widetilde{\Omega}_m = \langle h_t h^{-1}, \sigma\rangle = \langle -[q_t, q] + mq, \sigma\rangle \tag{21}$$

is a conserved quantity of our magnetic pendulum. This integral is a perturbation of the angular momentum $\langle [q_t, q], \sigma\rangle$ of the pendulum with respect to the axis of gravitation. The perturbation term $m\langle q, \sigma\rangle$ stems from the presence of the magnetic monopole.

Let now

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau)] \tag{22}$$

be the Maxwell–Bloch equation in which τ is a *constant* element of $\mathfrak{su}(2)$. Our first family of solutions describes the waves whose constituent oscillators are the charged spherical pendula in the field of a magnetic monopole. Let

$$g(t, x) = h(kx - \omega t) = h(s)$$

take values in $SU(2)$. Then

$$(g_t g^{-1})_t + c(g_t g^{-1})_x = (\omega^2 - k\omega c)(h_s h^{-1})_s.$$

The map $g(t, x)$ solves the Maxwell–Bloch equation (30) if and only if $h(s)$ is a solution of the C Neumann equation

$$(h_s h^{-1})_s = \left[\left(\frac{1}{\omega^2 - \omega k c}\right)\sigma, \mathrm{Ad}_h(\tau)\right]. \tag{23}$$

It is important to note that the solutions $g(t,x) = h(kx - \omega t)$ indeed satisfy the constraint $\langle g_t g^{-1}, \sigma\rangle = \text{const}$. This is insured by the fact that (21) is a conserved quantity. Let us express the solution $g(t,x) = h(kx - \omega t)$ in terms of the original physical quantities of the Maxwell–Bloch equations, namely in terms of the electrical field E, the polarization of the medium P and the level inversion D. To this end, it is better to use an appropriate solution of a magnetic spherical pendulum. If $h(s)$ is a solution of (23), then $q(s) = \mathrm{Ad}_{h(s)}(\tau)\colon I \to S_\tau \subset \mathfrak{su}(2)$ is an evolution of our pendulum. Let us denote

$$q(s) = \begin{pmatrix} \mathrm{i}q_3(s) & q_1(s) + \mathrm{i}q_2(s) \\ -q_1(s) + \mathrm{i}q_2(s) & -\mathrm{i}q_3(s) \end{pmatrix} = \mathrm{Ad}_{h(s)}(\tau)\colon I \longrightarrow S^2_\tau \subset \mathfrak{su}(2) \cong \mathbb{R}^3 \tag{24}$$

and let

$$\Omega_1(s) = q_2(s)\dot{q}_3(s) - q_3(s)\dot{q}_2(s), \qquad \Omega_2(s) = q_3(s)\dot{q}_1(s) - q_1(s)\dot{q}_3(s)$$

be the components of the angular momentum with respect to the two directions perpendicular to gravity. Formulae (6), (8), (20), (22) and (23) now yield the proof of the following proposition.

Proposition 3. *Let*

$$(q_1(s), q_2(s), q_3(s)) : I \longrightarrow S_\tau \subset \mathfrak{su}(2) = \mathbb{R}^3$$

be a solution of the magnetic spherical pendulum with charge m, and the gravitational potential equal to

$$V(q) = \left(\frac{1}{\omega^2 - \omega k c}\right)\langle\sigma, q\rangle.$$

The functions

$$\begin{aligned} E(t,x) &= (\Omega_1 - mq_1)(\omega t - kx) + \mathrm{i}(\Omega_2 - mq_2)(\omega t - kx), \\ P(t,x) &= q_1(\omega t - kx) + \mathrm{i}q_2(\omega t - kx)), \\ D(t,x) &= q_3(\omega t - kx) \end{aligned}$$

solve the Maxwell–Bloch equations (30).

The above solutions describe a family of non-linear travelling waves. The constituent oscillators of these waves are the magnetic spherical pendula in the same way as the harmonic oscillators are the constituent oscillators of the harmonic waves. The phase velocity ω/k of our waves increases with the increasing gravitational potential $V(q)$. When $V(q)$ approaches infinity, the velocity of the waves approaches the speed of light c in the medium.

Now we shall show that the famous 2π-pulse solution of the theory of self-induced transparency appears as a special case of the family described above. Let us consider the symplectic quotient of our C Neumann system at the zero value of the moment map μ given by (19). In this case, the reduced system is the usual spherical pendulum $(T^*S^2, \omega_{\mathrm{c}}, H_{\mathrm{sp}})$ without the magnetic monopole. The conserved quantities of this system are the energy H_{sp} and the angular momentum $\widetilde{\Omega}(q, q_t) = \langle[q_t, q], \sigma\rangle$ with respect to the axis of gravitation. If we have $\widetilde{\Omega}(q, q_t) = 0$, this system is reduced to the usual planar gravitational pendulum. Without the loss of generality, we can take $\tau = \sigma$ and confine the path q given by (24) to the circle

$$q(s) = \begin{pmatrix} \mathrm{i}q_3(s) & \mathrm{i}q_2(s) \\ \mathrm{i}q_2(s) & -\mathrm{i}q_3(s) \end{pmatrix} : I \longrightarrow S^1 \subset S^2_\sigma \subset \mathfrak{su}(2) \cong \mathbb{R}^3.$$

If we parametrize this circle by the angle $\frac{\theta}{2}$, we get the path

$$q(\theta(s)) = \mathrm{Ad}_{h(\theta(s))} = \begin{pmatrix} \mathrm{i}\cos 2\theta(s) & \mathrm{i}\sin 2\theta(s) \\ \mathrm{i}\sin 2\theta(s) & -\mathrm{i}\cos 2\theta(s) \end{pmatrix} : I \longrightarrow S^1. \tag{25}$$

In this case, the suitable lift $h(\theta(s))\colon I \to SU(2)$ is clearly given by

$$h(s) = \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix} : I \longrightarrow U(1) \subset SU(2)$$

and thus

$$h_s h^{-1}(s) = \begin{pmatrix} 0 & \theta'(s) \\ \theta'(s) & 0 \end{pmatrix} : I \longrightarrow \mathfrak{u}(1). \tag{26}$$

Recall that $g(t,x)\colon I \times \mathbb{R} \to U(1) \subset SU(2)$ is a solution of the Maxwell–Bloch equation if $g(t,x) = h(kx - \omega t)$ and $h(s)$ is a solution of the suitable C Neumann oscillator. Let $\theta(s)\colon I \to \mathbb{R}$ be a solution of the gravitational pendulum whose potential is equal to

$$V(\theta) = -\kappa^2 \cos\theta = \left(\frac{1}{\omega^2 - \omega k c}\right)\cos\theta.$$

Then

$$E(t,x) = \theta'(\omega t - kx), \qquad P(t,x) = \sin\theta(\omega t - kx), \qquad D(t,x) = \cos\theta(\omega t - kx) \tag{27}$$

is a solution of the Maxwell–Bloch equations. This can be seen from equations (6), (8), (25) and (26).

The gravitational pendulum has a well-known homoclinic solution which corresponds to the energy the pendulum has at the unstable equilibrium (when it is at rest on the top of the circle). In other words, this is the solution that travels along the separatrix in the phase portrait of the pendulum. It is well known and indeed not difficult to see that this solution is given by

$$\theta(s) = 4\arctan\left(\mathrm{e}^{\kappa s}\right) - \pi.$$

For the calculation see e.g. [22]. If we now put this solution into (27), we finally get the 2π-pulse solitonic solution

$$\begin{aligned} E(t,x) &= 2\kappa\,\mathrm{sech}(\kappa(\omega t - kx)), \\ P(t,x) &= \sin(4\arctan\left(\mathrm{e}^{2\kappa(\omega t - kx)}\right) - \pi), \\ D(t,x) &= \cos\left(4\arctan(\mathrm{e}^{2\kappa(\omega t - kx)}) - \pi\right). \end{aligned}$$

Remark 1. We note that the above construction of the solutions which stems from the planar gravitational pendulum corresponds to the well-known reduction of the Maxwell–Bloch equations to the sine-Gordon equation.

Our second family of solutions is simpler and it is obtained by the ansatz

$$g(t,x) = u(t,x)\cdot h(t)\colon I \times S^1 \to SU(2),$$

where $h(t)\colon I \to SU(2)$ solves the C Neumann system $(T^*SU(2), \omega_{\mathrm{c}}, H_{\mathrm{cn}})$. If we insert this into (22) and if we take into account that $h(t)$ solves $(h_t h^{-1})_t = [\sigma, \mathrm{Ad}_h(\tau)]$, we see that $u(t,x)$ must commute with σ and that it satisfies the equation

$$(u_t u^{-1})_t + c(u_t u^{-1})_x + [u_t u^{-1} + c u_x u^{-1}, \mathrm{Ad}_u(h_t h^{-1})] = 0.$$

Commutation of u with σ gives $u(t,x) = \mathrm{Exp}(f(t,x)\cdot\sigma)$ for some function $f(t,x)$: $I \times S^1 \to SU(2)$. From the above equation, we get the following one for f:

$$(f_{tt} + c f_{tx})\cdot\sigma + (f_t + c f_x)\cdot[\sigma, h_t h^{-1}] = 0.$$

The elements σ and $[\sigma, h_t h^{-1}]$ are orthogonal with respect to the Killing form on $\mathfrak{su}(2)$; therefore, we simply have $f_t + c f_x = 0$. This is the 'outgoing part' of the wave equation and its

D'Alembert solutions are of the form $f(t,x) = w(\omega t - kx)$, where $w\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function of one variable. Thus, the mapping

$$g(t,x) = \mathrm{Exp}(w(\omega t - kx)\cdot\sigma)\cdot h(t) : I \times S^1 \longrightarrow SU(2) \tag{28}$$

is a solution of equation (22) for arbitrary function w and for every solution $h(t)$ of our C Neumann system. In $g(t,x)$, the solution $h(t)$ of the C Neumann system is rotated in the vertical direction of the Hopf fibration $S^1 \hookrightarrow SU(2) \to S^2_\tau$, given by the projection $h \mapsto \mathrm{Ad}_h(\tau)$. Rotation is caused by a harmonic wave which travels with the speed of light $c = \omega/k$. In the case when $\langle g_t g^{-1}, \sigma\rangle = \text{const}$, which corresponds to the Maxwell–Bloch equations (4), we simply have

$$w(\omega t - kx) = \omega t - kx + a, \tag{29}$$

where a is a constant. Then the above discussion and expressions (6), (8), (28) and (29) in which we neglect the inessential phase shift a, give us the following result.

Proposition 4. *Let*

$$E_t + cE_x = P, \qquad P_t = ED - \beta P, \qquad D_t = -\tfrac{1}{2}(\overline{E}P + E\overline{P}) \tag{30}$$

be the Maxwell–Bloch equations. The functions

$$E(t,x) = \mathrm{e}^{\mathrm{i}2(\omega t - kx)} 2((\Omega_1(t) - mq_1(t)) + \mathrm{i}(\Omega_2(t) - mq_2(t))),$$
$$P(t,x) = \mathrm{e}^{\mathrm{i}2(\omega t - kx)}(q_1(t) + \mathrm{i}q_2(t)),$$
$$D(t,x) = q_3(t)$$

solve (30) for every solution

$$(q_1(t), q_2(t), q_3(t)) : I \longrightarrow S^2_\tau \subset \mathbb{R}^3$$

of the magnetic spherical pendulum with the charge equal to m. For the longitudinal relaxation rate β, we have the expression

$$\beta = \widetilde{\Omega}_{\mathrm{m}} - c,$$

where $\widetilde{\Omega}_{\mathrm{m}}$ is the value of the integral (21) along our chosen solution $(q_1(t), q_2(t), q_3(t))$ of the magnetic spherical pendulum.

5. Hamiltonian structure with the canonical symplectic form

As we stressed above, in the Hamiltonian system $(T^*LSU(2), \omega_{\mathrm{c}} + c\omega_{\mathrm{m}}, H_{\mathrm{mb}})$ the canonical symplectic structure ω_{c} on $T^*LSU(2)$ is perturbed by the 2-form ω_{m}. Let $(T^*N, \omega_{\mathrm{c}} + \sigma_{\mathrm{m}}, H)$ be a Hamiltonian system, where ω_{c} is the canonical structure on T^*N and σ_{m} is the pull-back of some 2-form Σ_{m} on N. Being closed, the form Σ_{m} is locally exact, $(\Sigma_{\mathrm{m}})_q = \mathrm{d}\theta_q$. Then (again locally) a path $q(t)\colon I \to N$ is a solution of the Hamiltonian system $(T^*N, \omega_{\mathrm{c}} + \sigma_{\mathrm{m}}, H)$ if and only if it is a solution of the system $(T^*N, \omega_{\mathrm{c}}, H_s)$, where the Hamiltonian function $H_s\colon T^*N \to \mathbb{R}$ is given by the formula $H_s(q, p_q) = H(q, p_q + \theta_q)$. For the proof see [10], p 158. This shows that the magnetic terms are responsible for forces which depend linearly on the momentum. The geometrization of such forces is provided by the Kaluza–Klein theory, as mentioned in the introduction.

First we shall describe the Kaluza–Klein geometrization in general. We have to consider the magnetic terms which can be topologically non-trivial, since this is the case in the Maxwell–Bloch system.

We recall the statement of Weil's theorem. Let N be a manifold and let $\Sigma_{\mathrm{m}} \in \Omega^2(N)$ be an integral 2-form. This means that for every 2-cycle S in N the value of the pairing $\int_S \Sigma_{\mathrm{m}}$ is an integer. Weil's theorem then ensures the existence of the circle bundle $\phi\colon M \to N$ equipped with the connection θ, such that the curvature F_θ is precisely the 2-form Σ_{m}. Proof of Weil's theorem can be found in many texts about the geometric quantization, e.g. in [23].

Weil's connection θ on M decomposes the tangent bundle T_qM into the horizontal and the vertical part, $T_qM = \mathrm{Hor}_q \oplus \mathrm{Vert}_q$. This decomposition induces the decomposition of the cotangent space

$$T_q^*M = \mathrm{Hor}_q^* \oplus \mathrm{Vert}_q^*. \tag{31}$$

Note that $\mathrm{Hor}_q^* = Ann(\mathrm{Vert}_q)$ and $\mathrm{Vert}_q^* = Ann(\mathrm{Hor}_q)$, where Ann is the annihilator. Let $\phi^* : T^*_{\phi(q)}N \longrightarrow \mathrm{Hor}_q^*$ be the adjoint of the derivative $(D\phi)_q\colon T_qM \to T_qN$ restricted to Hor_q. The map ϕ^* is of course an isomorphism. Let us define the lifted Hamiltonian $\widetilde{H}$ on T^*M by the formula

$$\widetilde{H}(q, p_q) = H(\phi(q), (\phi^*)^{-1}(\mathrm{Hor}^*(p_q))) + (\mathrm{Vert}_q^*(p_q))^2. \tag{32}$$

The natural $U(1)$-action on M lifts to the action $\rho : U(1) \times T^*M \longrightarrow T^*M$ which is Hamiltonian with respect to the canonical structure ω_{c} on T^*M. Let $\mu : T^*M \longrightarrow \mathfrak{u}(1) = \mathrm{i}\mathbb{R}$ be the moment map of ρ. Weil's theorem enables us to state the following claim.

Proposition 5. *Let $(T^*N, \omega_{\mathrm{c}} + \sigma_{\mathrm{m}}, H)$ be a Hamiltonian system and let the magnetic term Σ_{m} be an integral 2-form on N. Then the Hamiltonian system $(T^*M, \Omega_{\mathrm{c}}, \widetilde{H})$ whose symplectic structure Ω_{c} is canonical and whose Hamiltonian $\widetilde{H}$, given by (32), is invariant with respect to the action ρ. Its symplectic reduction $(\mu^{-1}(\mathrm{i}a)/U(1), \omega_{\mathrm{sq}}, H_r)$ is the original system $(T^*N, \omega_{\mathrm{c}} + a\sigma_{\mathrm{m}}, H)$.*

Proof. The invariance of $\widetilde{H}$ with respect to the action ρ is a direct consequence of the fact that the connection θ is invariant with respect to ρ.

Whenever the action on the cotangent bundle is lifted from the action on the base space, the moment map $\mu\colon T^*M \to \mathrm{i}\mathbb{R}$ is given by $\mu(q, p_q)(\xi) = p_q(\xi_N)$, where ξ_N is the infinitesimal action on the base space. In our case, this gives $\mu(q, p_q) = p_q^{\mathrm{V}}$, where $p_q^{\mathrm{V}} = \mathrm{Vert}_q^*(p_q)$ is the vertical part of the decomposition $p_q = p_q^{\mathrm{H}} + p_q^{\mathrm{V}}$ given by (31). This shows that $\widetilde{H}$ induces the function $H + a^2$ on the symplectic quotient $\mu^{-1}(\mathrm{i}a)/U(1)$. This function differs from our original Hamiltonian by an irrelevant constant.

Now we have to prove that the symplectic quotient of $(T^*M, \Omega_{\mathrm{c}})$ is indeed $(T^*N, \omega_{\mathrm{c}} + a\sigma_{\mathrm{m}})$. Let $\vartheta \in \Omega^1(T^*M)$ be the tautological 1-form. Then $\mathrm{d}\vartheta = \Omega_{\mathrm{c}}$. For every pair of tangent vectors $X_{(q,p_q)}, Y_{(q,p_q)} \in T_{(q,p_q)}(T^*M)$, the well-known formula for the derivative of 1-forms gives

$$(\Omega_{\mathrm{c}})_{(q,p_q)}(X_{(q,p_q)}, Y_{(q,p_q)}) = (\hat{X}(\vartheta(\hat{Y}) - \hat{Y}(\vartheta(\hat{X}) - \vartheta([\hat{X}, \hat{Y}]))\,|_{(q,p_q)}, \tag{33}$$

where $\hat{X}, \hat{Y}$ are the arbitrary vector fields in a neighbourhood of (q, p_q) which extend our tangent vectors. Choose a local trivialization of $T(T^*M)$ and denote $X_{(q,p_q)} = (X_{\mathrm{b}}, X_{\mathrm{ct}})$, where $X_{\mathrm{b}} \in T_qM$ and $X_{\mathrm{ct}} \in T_q^*M$. The tautological form is defined by $\vartheta_{(q,p_q)}(X_{\mathrm{b}}, X_{\mathrm{ct}}) = p_q(X_{\mathrm{b}})$. We can decompose it into the horizontal and the vertical part, $\vartheta = \vartheta^{\mathrm{H}} + \vartheta^{\mathrm{V}}$, by putting

$$\vartheta^{\mathrm{H}}_{(q,p_q)}(X_{\mathrm{b}}, X_{\mathrm{ct}}) = p_q^{\mathrm{H}}(X_{\mathrm{b}}), \qquad \vartheta^{\mathrm{V}}_{(q,p_q)}(X_{\mathrm{b}}, X_{\mathrm{ct}}) = p_q^{\mathrm{V}}(X_{\mathrm{b}}),$$

where $p_q^{\mathrm{H}} \in \mathrm{Hor}_q^*$ and $p_q^{\mathrm{V}} \in \mathrm{Vert}_q^*$.

Let us choose the extension vector field $\hat{X}$ of $X_{(q,p_q)} = (X_{\mathrm{b}}, X_{\mathrm{ct}})$ defined in some neighbourhood of (q, p_q) in the following way: decompose first $X_{\mathrm{b}} = X_{\mathrm{b}}^{\mathrm{H}} + X_{\mathrm{b}}^{\mathrm{V}}$ into the horizontal and the vertical part. Choose a vector field extending $(D\phi)_q\big(X_{\mathrm{b}}^{\mathrm{H}}\big)$ on N and let $\hat{X}_{\mathrm{b}}^{\mathrm{H}}$

be its unique $U(1)$-invariant horizontal lift. The stipulation for the extension $\hat{X}_{\mathrm{b}}^{\mathrm{V}}$ of $X_{\mathrm{b}}^{\mathrm{V}}$ is the following: the restriction of the function $p_q^{\mathrm{V}}\big(X_q^{\mathrm{V}}\big)$ on $\mu^{-1}(\mathrm{i}a) \subset T^*M$ must be constant. Let now $\hat{X}_{\mathrm{b}} = \hat{X}_{\mathrm{b}}^{\mathrm{H}} + \hat{X}_{\mathrm{b}}^{\mathrm{V}}$. Define the field $\hat{X}_{\mathrm{ct}}^{\mathrm{H}}$ analogously to the definition of $\hat{X}_{\mathrm{b}}^{\mathrm{H}}$ using the isomorphism ϕ^*, let $X_{\mathrm{ct}}^{\mathrm{V}}$ be an arbitrary vertical extension of $X_{\mathrm{ct}}^{\mathrm{V}}$ and let finally $\hat{X}_{\mathrm{ct}} = \hat{X}_{\mathrm{ct}}^{\mathrm{H}} + \hat{X}_{\mathrm{ct}}^{\mathrm{V}}$. We construct $\hat{Y}$ in the same manner as $\hat{X}$. Then we have

$$\big[\hat{X}_{\mathrm{b}}^{\mathrm{V}}, \hat{Y}_{\mathrm{b}}^{\mathrm{V}}\big] = 0 \qquad \text{and} \qquad [\hat{Y}_{\mathrm{b}}, \hat{X}_{\mathrm{b}}] = \big[\hat{Y}_{\mathrm{b}}^{\mathrm{H}}, \hat{X}_{\mathrm{b}}^{\mathrm{H}}\big]. \tag{34}$$

The first equation is obvious. For the second, denote by $\Phi(s)$ the flow of the vector field $Y_{\mathrm{b}}^{\mathrm{V}}$ and by $\varphi(s)$ the integral curve of $Y_{\mathrm{b}}^{\mathrm{V}}$ beginning at q. Then

$$\big[\hat{Y}_{\mathrm{b}}^{\mathrm{V}}, \hat{X}_{\mathrm{b}}^{\mathrm{H}}\big] = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (D_{\varphi(s)}(\Phi^{-1}(s))\big(\hat{X}_{\mathrm{b}}^{\mathrm{H}}(\varphi(s))\big) = 0,$$

since $\hat{X}_{\mathrm{b}}^{\mathrm{H}}$ is $U(1)$-invariant. The second equation of (34) now follows immediately.

From our construction of the fields $\hat{X} = \hat{X}^{\mathrm{H}} + \hat{X}^{\mathrm{V}}$ and $\hat{Y} = \hat{Y}^{\mathrm{H}} + \hat{Y}^{\mathrm{V}}$, it also follows:

$$\hat{X}^{\mathrm{V}}\big(p_q^{\mathrm{H}}\big(\hat{Y}_{\mathrm{b}}^{\mathrm{H}}\big)\big) = 0, \qquad \hat{X}^{\mathrm{H}}\big(p_q^{\mathrm{V}}\big(\hat{Y}_{\mathrm{b}}^{\mathrm{V}}\big)\big) = 0 \quad \text{on} \quad \mu^{-1}(\mathrm{i}a). \tag{35}$$

The first equation is true because the function $p_q^{\mathrm{H}}\big(\hat{Y}_{\mathrm{b}}^{\mathrm{H}}\big)$ is invariant with respect to the action ρ and the field $\hat{X}^{\mathrm{V}}$ is collinear with the infinitesimal action of ρ. The second follows from the fact that $p_q^{\mathrm{V}}\big(\hat{Y}_{\mathrm{b}}^{\mathrm{V}}\big)$ is constant on $\mu^{-1}(\mathrm{i}a)$. We can express ϑ^{H} and ϑ^{V} slightly more explicitly:

$$\vartheta_{(q,p_q)}^{\mathrm{H}}(X_{\mathrm{b}}, X_{\mathrm{ct}}) = p_q^{\mathrm{H}}\big(X_{\mathrm{b}}^{\mathrm{H}}\big), \qquad \vartheta_{(q,p_q)}^{\mathrm{V}}(X_{\mathrm{b}}, X_{\mathrm{ct}}) = p_q^{\mathrm{V}}\big(X_{\mathrm{b}}^{\mathrm{V}}\big). \tag{36}$$

Define the projection map $\Psi\colon T^*M \to T^*N$ by

$$\Psi(q, p_q) = \big(\phi(q), (\phi^*)^{-1}\big(p_q^{\mathrm{H}}\big)\big),$$

where ϕ^* is again the adjoint of the derivative $D_q\phi$ restricted to $\mathrm{Hor}_q \subset T_qM$. Formulae (33), (34), (35) and (36) now give

$$\mathrm{d}\vartheta_{(q,p_q)}^{\mathrm{H}}\big(X_{(q,p_q)}, Y_{(q,p_q)}\big) = (\Psi^*(\omega_{\mathrm{c}}))_{(q,p_q)}\big(X_{(q,p_q)}, Y_{(q,p_q)}\big)$$

and

$$\mathrm{i}^*\big(\mathrm{d}\vartheta_{(q,p_q)}^{\mathrm{V}}\big)\big(X_{(q,p_q)}, Y_{(q,p_q)}\big) = \big(\mathrm{i}p_q^{\mathrm{V}}\big) \cdot ((\Psi^*)(\sigma_{\mathrm{m}}))_{(q,p_q)}\big(X_{(q,p_q)}, Y_{(q,p_q)}\big),$$

where $\mathrm{i}\colon \mu^{-1}(\mathrm{i}a) \to T^*M$ is the inclusion. Here we have used the fact that Σ_{m} is the curvature of the connection θ and is therefore given by $(\Sigma_{\mathrm{m}})_q(X_q, Y_q) = \mathrm{Vert}_q([\mathrm{Hor}(\widetilde{X}), \mathrm{Hor}(\widetilde{Y})]_q)$, where $\widetilde{X}$ and $\widetilde{Y}$ are arbitrary vector fields on M extending $X_q, Y_q \in T_qM$. Note that $\sigma_{\mathrm{m}} = \pi^*(\Sigma_{\mathrm{m}})$ and that $\big(\mathrm{i}p_q^{\mathrm{V}}\big) = a$ is a real number.

Recall now that $\Omega_{\mathrm{c}} = \mathrm{d}\vartheta = \mathrm{d}\vartheta^{\mathrm{H}} + \mathrm{d}\vartheta^{\mathrm{V}}$. The above expressions show that for every $\mathrm{i}a \in \mathfrak{u}(1)$, the pull-back $\mathrm{i}^*(\Omega_{\mathrm{c}})$ via the inclusion map $\mathrm{i}\colon \mu(\mathrm{i}a)^{-1} \to T^*M$ satisfies the relation

$$\mathrm{i}^*(\Omega_{\mathrm{c}}) = \Psi^*(\omega_{\mathrm{c}} + a\sigma_{\mathrm{m}}).$$

Finally, we note that the natural projection $\Pi\colon T^*M \to T^*N$ of the action ρ is precisely the map Ψ. Therefore, the above formula completes the proof of the theorem. □

Now we shall describe the Kaluza–Klein expression for the Maxwell–Bloch system. It will be instructive to construct it directly, without referring to proposition 5.

The 2-form $\Omega_{\mathrm{m}} \in \Omega^2(LSU(2))$ plays an important role in the theory of the loop group $LSU(2)$. It is essentially the cocycle associated with the central extension $\widetilde{L}SU(2)$ of $LSU(2)$.

The central extension

$$\mathbb{R} \longrightarrow \widetilde{L}\mathfrak{su}(2) = L\mathfrak{su}(2) \oplus \mathbb{R} \longrightarrow L\mathfrak{su}(2)$$

of the Lie algebra $L\mathfrak{su}(2)$ is given by

$$[(\xi,\lambda),(\eta,\mu)] = \left([\xi,\eta], \frac{1}{2\pi}(\omega_{\mathrm{m}})_e(\xi,\eta)\right) = \left([\xi,\eta], -\frac{1}{2\pi}\int_{S^1}\langle \xi_x,\eta\rangle\,\mathrm{d}x\right). \tag{37}$$

Since the skew form $\frac{1}{2\pi}(\omega_{\mathrm{m}})_e$ is an *integral* cocycle on $L\mathfrak{su}(2)$, it defines the central extension

$$S^1 \longrightarrow \widetilde{L}SU(2) \xrightarrow{\phi} LSU(2) \tag{38}$$

on the group level. Geometrically, the central extension $\widetilde{L}SU(2)$ is the $U(1)$ principal bundle over $LSU(2)$, equipped with a right-invariant connection θ whose value at the identity $e \in \widetilde{L}SU(2)$ is given by

$$\theta(\widetilde{X}) = \theta(X,x) = x, \qquad \widetilde{X} \in T_e\widetilde{L}SU(2) = \widetilde{L}\mathfrak{su}(2) = L\mathfrak{su}(2) \oplus \mathrm{i}\mathbb{R}.$$

Alternatively, the connection θ is given by the right-invariant distribution in $T\widetilde{L}SU(2)$. At the identity $e \in \widetilde{L}SU(2)$, it is given by

$$T_e\widetilde{L}SU(2) = \widetilde{L}\mathfrak{su}(2) = L\mathfrak{su}(2) \oplus \mathbb{R} = (\mathrm{Hor}_\theta)_e \oplus (\mathrm{Vert}_\theta)_e.$$

The curvature of θ is equal to the 2-form $\mathrm{i}\Omega_{\mathrm{m}}$.

Let us denote by $\rho: U(1) \times T^*\widetilde{L}SU(2) \to T^*\widetilde{L}SU(2)$ the cotangent lift of the natural $U(1)$-action. We note that we only need the expression of the infinitesimalization at the identity $e \in \widetilde{L}SU(2)$ of this action. However, the reader can easily find the formula for the entire action on $\widetilde{L}SU(2)$ from the information given in [24].

Clearly, ρ preserves the canonical symplectic structure Ω_{c} on $T^*\widetilde{L}SU(2)$ and is therefore Hamiltonian. The moment map $\mu : T^*\widetilde{L}SU(2) \longrightarrow \mathrm{i}\mathbb{R}$ is given by $\mu(\widetilde{g}, p_{\widetilde{g}}) = p_{\widetilde{g}}(\xi_\rho)$, where the vector field ξ_ρ is the infinitesimal action on the base space $\widetilde{L}SU(2)$. Let us trivialize the tangent and cotangent bundles of $\widetilde{L}SU(2)$ by the right translations. Then for every $\widetilde{g}$ we have $T_{\widetilde{g}}\widetilde{L}SU(2) \cong L\mathfrak{su}(2) \oplus \mathrm{i}\mathbb{R}$ and $T^*_{\widetilde{g}}\widetilde{L}SU(2) \cong (L\mathfrak{su}(2) \oplus \mathrm{i}\mathbb{R})^*$. Under this identification, we have $p_{\widetilde{g}} = (p_g, \psi)$, $\xi_\rho = (0, 1)$ and therefore

$$\mu(\widetilde{g}, p_{\widetilde{g}}) = \psi.$$

Now we shall decompose the canonical symplectic structure Ω_{c} on $T^*\widetilde{L}SU(2)$ with respect to the natural connection θ on the circle bundle $\widetilde{L}SU(2)$. We shall apply formula (12) for the canonical form on the cotangent bundle over a Lie group to the case when the Lie group is the central extension $\widetilde{L}SU(2)$. In the right trivialization, an element $(\widetilde{X}_{\mathrm{b}}, \widetilde{X}_{\mathrm{ct}}) \in T_{(\widetilde{g},p_{\widetilde{g}})}(T^*\widetilde{L}SU(2)) = \widetilde{L}\mathfrak{su}(2) \times (\widetilde{L}\mathfrak{su}(2))^*$ has the form

$$(\widetilde{X}_{\mathrm{b}}, \widetilde{X}_{\mathrm{ct}}) = ((X_{\mathrm{b}}, x_{\mathrm{b}}), (X_{\mathrm{ct}}, x_{\mathrm{ct}})), \qquad X_{\mathrm{b}} \in L\mathfrak{su}(2), \quad X_{\mathrm{ct}} \in (L\mathfrak{su}(2))^*, \quad x_{\mathrm{b}}, x_{\mathrm{ct}} \in \mathbb{R}.$$

Formula (12) and the Lie algebra bracket (37) of the central extension then give

$$\begin{aligned}(\Omega_{\mathrm{c}})_{(\widetilde{g},p_{\widetilde{g}})} &= -\langle X_{\mathrm{ct}}, Y_{\mathrm{b}}\rangle + \langle Y_{\mathrm{ct}}, X_{\mathrm{b}}\rangle + \langle p_g, [X_{\mathrm{b}}, Y_{\mathrm{b}}]\rangle \\ &\quad - x_{\mathrm{ct}} y_{\mathrm{b}} + y_{\mathrm{ct}} x_{\mathrm{b}} - \psi \cdot \frac{1}{2\pi}\int_{S^1}\langle (X_{\mathrm{b}})_x, Y_{\mathrm{b}}\rangle\,\mathrm{d}x,\end{aligned}$$

where $p_{\widetilde{g}} = (p_g, \psi) \in (L\mathfrak{su}(2) \oplus \mathbb{R})^*$. Let the projection map $F: T^*\widetilde{L}SU(2) \to T^*LSU(2)$ in the right trivializations be given by $F(\widetilde{g}, p_{\widetilde{g}}) = F(\widetilde{g}, (p_g, \psi)) = (\phi(g), p_g)$. The above formulae give

$$(\Omega_{\mathrm{c}})_{(\widetilde{g},p_{\widetilde{g}})} = F^*(\omega_{\mathrm{c}})_{(\widetilde{g},p_{\widetilde{g}})} + (\omega_{\mathrm{fib}})_{(\widetilde{g},p_{\widetilde{g}})} + \psi \cdot F^*(\omega_{\mathrm{m}})_{(\widetilde{g},p_{\widetilde{g}})}. \tag{39}$$

Here ω_{c} is the canonical structure on $T^*LSU(2)$. The second term ω_{fib} is the canonical cotangent form on the fibre of the map F. For every $(g, p_g) \in T^*LSU(2)$, the fibre $F^{-1}(g, p_g)$ is the cotangent bundle T^*S^1 over the circle. Finally, $F^*(\omega_{\mathrm{m}})$ is the pull-back of the curvature

ω_{m} of the connection θ on $\widetilde{L}SU(2) \to LSU(2)$. Recall that ω_{m} is also the perturbation form in the Maxwell–Bloch Hamiltonian system.

Consider now the symplectic quotient of $T^*\widetilde{L}SU(2)$ with respect to the action ρ. Let ω_{sq} denote the induced symplectic structure on the symplectic quotient $\mu^{-1}(\psi)/U(1)$. The decomposition (39) proves the following result.

Proposition 6. *Let $\mu: T^*\widetilde{L}SU(2) \to \mathbb{R}$ be the moment map of the natural action $\rho: U(1) \times T^*\widetilde{L}SU(2) \to T^*\widetilde{L}SU(2)$. Then for the symplectic quotient $(\mu^{-1}(\psi)/U(1), \omega_{\mathrm{sq}})$ of $(T^*\widetilde{L}SU(2), \Omega_{\mathrm{c}})$, we have*

$$(\mu^{-1}(\psi)/U(1), \omega_{\mathrm{sq}}) = (T^*LSU(2), \omega_{\mathrm{c}} + \psi\omega_{\mathrm{m}}).$$

The above proposition gives us now the expression of the Maxwell–Bloch Hamiltonian system in terms of a canonical symplectic structure.

Theorem 2. *Let $(T^*\widetilde{L}SU(2), \Omega_{\mathrm{c}}, \widetilde{H})$ be the Hamiltonian system on $T^*\widetilde{L}SU(2)$, where Ω_{c} is the canonical cotangent symplectic structure and the function $\widetilde{H}_{\mathrm{mb}} : T^*\widetilde{L}SU(2) \longrightarrow \mathbb{R}$ is given by the formula*

$$\widetilde{H}_{\mathrm{mb}}(\widetilde{g}, p_{\widetilde{g}}) = \tfrac{1}{2}\|p_{\widetilde{g}}\|^2 + \langle\langle \sigma, \mathrm{Ad}_{\widetilde{g}}(\tau)\rangle\rangle,$$

with $\sigma = \frac{1}{2}\mathrm{diag}(\mathrm{i}, -\mathrm{i}) \in \mathfrak{su}(2)$ and $\tau \in L\mathfrak{su}(2)$ an arbitrary loop. Then the moment map $\mu: T^*\widetilde{L}SU(2) \to \mathbb{R}$ of the $U(1)$-action ρ is an integral of the system $(T^*\widetilde{L}SU(2), \Omega_{\mathrm{c}}, \widetilde{H}_{\mathrm{mb}})$. For the reduced Hamiltonian system, we have

$$(\mu^{-1}(\psi)/U(1), \omega_{\mathrm{sq}}, H_{\mathrm{sq}}) = (T^*LSU(2), \omega_{\mathrm{c}} + \psi\omega_{\mathrm{m}}, H_{\mathrm{mb}}),$$

where $(T^*LSU(2), \omega_{\mathrm{c}} + \psi\omega_{\mathrm{m}}, H)$ is the system whose equation of motion is

$$(g_t g^{-1})_t + \psi(g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau)].$$

When $\psi = c$, this is precisely the Maxwell–Bloch equation.

Remark 2. The Kaluza–Klein charge of the additional degree of freedom in $\widetilde{L}SU(2)$ is ψ. We can write the above equation in the form

$$(g_t g^{-1})_t(x) = \psi\frac{1}{\epsilon}(g_t g^{-1}(x-\epsilon) - g_t g^{-1}(x+\epsilon))|_{\epsilon\to 0} + [\sigma, \mathrm{Ad}_{g(x)}(\tau(x))].$$

This shows that the charge ψ is the strength of the magnetic interaction between the neighbouring C Neumann oscillators in the chain. An even clearer description says that the momentum π is equal to the speed of light in the medium. The fact that π is an integral of the extended system $(T^*\widetilde{L}SU(2), \Omega_{\mathrm{c}}, \widetilde{H})$ coincides with the fundamental physical law which says that the speed of light $\psi = c$ in the medium is constant.

Proof of theorem 2. We only have to check that the Hamiltonian $\widetilde{H}$ is invariant with respect to the $U(1)$-action ρ. For the kinetic energy, we have

$$\|p_{\widetilde{g}}\|^2 = \|(p_g, \psi)\|^2 = \|p_g\|^2 + \psi^2,$$

which is clearly invariant. In the potential energy term, we have the adjoint action of $\widetilde{L}SU(2)$ on an element from $(\widetilde{L}\mathfrak{su}(2))$. The adjoint action is given by the formula

$$\mathrm{Ad}_{\widetilde{g}}(\widetilde{\beta}) = \mathrm{Ad}_{\phi(\widetilde{g})}(\beta, b) = \left(\mathrm{Ad}_g(\beta), b - \frac{1}{2\pi}\int_{S^1}\langle g^{-1}g_x, \beta\rangle\,\mathrm{d}x\right).$$

This can be seen from the fact that the extension $\widetilde{L}SU(2)$ of $LSU(2)$ is central and from formula (37) for the Lie bracket in $\widetilde{L}\mathfrak{su}(2)$. The natural inclusion of the element $\sigma \in \mathfrak{su}(2)$

into the group $\widetilde{L}\mathfrak{su}(2)$ has the form $\mathrm{i}(\sigma) = (\sigma, 0) \in L\mathfrak{su}(2) \oplus \mathbb{R}$. Recall that the inner product on $\widetilde{L}\mathfrak{su}(2)$ is given by

$$\langle\!\langle (\alpha, a), (\beta, b) \rangle\!\rangle = \int_{S^1} \langle \alpha, \beta \rangle \, \mathrm{d}x + a \cdot b. \tag{40}$$

From this we see $\langle\!\langle \sigma, \mathrm{Ad}_{\widetilde{g}}(\tau(x)) \rangle\!\rangle = \int_{S^1} \langle \sigma, \mathrm{Ad}_{\phi(\widetilde{g})}(\tau(x)) \rangle \, \mathrm{d}x$. This expression is clearly invariant with respect to the action ρ. (The orbits of ρ are $\phi^{-1}(g)$.) The statement of the theorem now follows directly from proposition 6. □

In [26], the authors describe a Hamiltonian structure of the Maxwell–Bloch equation, but their structure is different from the one constructed above. A quick way to establish the nonequivalence of the two structures is to observe that the symplectic structure in [26] does not include the derivatives of the variables with respect to x co-ordinate, while our symplectic structure does. The fact that the Maxwell–Bloch equations are endowed with two nonequivalent Hamiltonian structures is of course very important. We intend to study this topic in another paper.

6. Lagrangian structure of the Maxwell–Bloch equations

In this section we shall investigate the Lagrangian structure of equation (9). To simplify the notation we put $c = 1$. The fact that the magnetic term $\omega_{\mathrm{m}} \in \widetilde{L}SU(2)$ is topologically non-trivial will play a crucial role. The Lagrangian expression of systems with non-trivial magnetic terms was studied by Novikov in [29]. Although we focus on the Maxwell–Bloch system, our construction of the Lagrangian formulation works for any Hamiltonian system with an integral non-trivial magnetic term. Our construction is different from the one described in [29]. The essential ingredient in our approach is the Kaluza–Klein extension, which makes the problem quite straightforward.

The Lagrangian expression of the Maxwell–Bloch equations on the original, non-extended configuration space $LSU(2)$ is more intricate, if less general. In particular, it works only for the temporally periodic solutions of the Maxwell–Bloch equations. It has essentially the same structure as the WZWN model which was introduced by Witten in [27, 28]. Again, our construction could be applied to arbitrary Hamiltonian systems with non-trivial magnetic terms.

We shall start by applying the Legendre transform to the Kaluza–Klein expression $(T^*\widetilde{L}SU(2), \Omega_{\mathrm{c}}, \widetilde{H}_{\mathrm{mb}})$ of the Maxwell–Bloch system. Let $T\widetilde{L}SU(2)$ be the tangent bundle. As before, we will work in the trivialization of $T\widetilde{L}SU(2)$ by the right translations. On the Lie algebra $\widetilde{L}\mathfrak{su}(2) = T_e\widetilde{L}SU(2)$, we have the inner product given by (40). By $\langle\!\langle -, - \rangle\!\rangle_{\widetilde{g}}$ we denote the value on $T_{\widetilde{g}}\widetilde{L}SU(2)$ of the right-invariant metric on $\widetilde{L}SU(2)$ whose value at the identity is given by (40). Note that the metric $\langle\!\langle -, - \rangle\!\rangle_{\widetilde{g}}$ is not bi-invariant, since the inner product (40) is not Ad-invariant. We can use our metric for the identification

$$T^*_{\widetilde{g}}\widetilde{L}SU(2) = \{p_{\widetilde{g}} \cdot (\widetilde{g}^{-1})^* = \langle\!\langle \widetilde{g}_t\widetilde{g}^{-1}, - \rangle\!\rangle, \qquad \widetilde{g}_t\widetilde{g}^{-1} \in T\widetilde{L}SU(2)\}.$$

Let now the Lagrangian $L\colon T\widetilde{L}SU(2) \to \mathbb{R}$ be given by

$$L(\widetilde{g}, \widetilde{g}_t) = \frac{1}{2}\langle\!\langle \widetilde{g}_t, \widetilde{g}_t \rangle\!\rangle_{\widetilde{g}} - \int_{S^1} \langle \sigma, \mathrm{Ad}_{\phi(\widetilde{g})}(\tau(x)) \rangle \, \mathrm{d}x. \tag{41}$$

In the right trivializations, the Legendre transformation $FL : T\widetilde{L}SU(2) \longrightarrow T^*\widetilde{L}SU(2)$ is given by $FL(\widetilde{g}_t\widetilde{g}^{-1}) = p_{\widetilde{g}} \cdot (\widetilde{g}^{-1})^* = \langle\!\langle \widetilde{g}_t\widetilde{g}^{-1}, - \rangle\!\rangle$. This gives us the following theorem.

Theorem 3. *Let the path* $\gamma(t) = (\widetilde{g}(t), p_{\widetilde{g}}(t)) : I \longrightarrow T^*\widetilde{L}SU(2)$ *be a solution of the Hamiltonian system* $(T^*\widetilde{L}SU(2), \Omega_c, \widetilde{H})$ *and let* $proj: T^*\widetilde{L}SU(2) \to \widetilde{L}SU(2)$ *be the natural projection. Then the path*

$$proj(\gamma(t)) = \widetilde{g}(t) : I \longrightarrow \widetilde{L}SU(2)$$

is an extremal of the Lagrangian functional

$$\mathcal{L}(\widetilde{g}(t)) = \int_I L(\widetilde{g}(t), \widetilde{g}_t(t))\,\mathrm{d}t, \tag{42}$$

where the function L is given by (41).

We shall now take a closer look at the *closed* extremals of the Lagrangian functional (42); that is, we will be interested in the loops $\widetilde{g}(t) : S^1 \longrightarrow \widetilde{L}SU(2)$ for which the value $\mathcal{L}(\widetilde{g}(t))$ is minimal. We shall see that the closed extremals of (42) can be characterized as the extremals of a Lagrangian functional on the non-extended loop group $LSU(2)$. But this Lagrangian will be of a non-standard kind in a similar way that the WZWN functional is. We will prove the following theorem.

Theorem 4. *Let* $g(t) : S^1 \longrightarrow LSU(2)$ *be a loop in* $LSU(2)$*. Let* $D \subset \mathbb{R}^2$ *be a disc whose boundary is our circle,* $\partial D = S^1$*, and let* $\hat{g} : D \longrightarrow \mathbb{R}$ *be an extension of g to the disc D. Then*

$$\mathcal{L}(g(t)) = \int_{S^1} \left(\frac{1}{2}\|g_t g^{-1}\|^2 - \langle\!\langle \sigma, \mathrm{Ad}_{g(t)}(\tau) \rangle\!\rangle \right) \mathrm{d}t + \int_{\hat{g}(D)} \omega_{\mathrm{m}} \tag{43}$$

is a well-defined map

$$\mathcal{L} : \{\text{Loops in } LSU(2)\} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1.$$

Furthermore, a loop $g(t): S^1 \to LSU(2)$ *is an extremal of* $\mathcal{L}$ *if and only if it is a solution of the Maxwell–Bloch equation*

$$(g_t g^{-1})_t + (g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau)].$$

Proof. The loop group $LSU(2)$ can be endowed with the structure of a Banach manifold in several different ways (see [24, 25]). Throughout this paper, we assume that $LSU(2)$ is equipped with a suitable Banach manifold structure which makes ω_{m} a *smooth* 2-form. Let $\{U_\alpha; \alpha \in A\}$ be an open covering of $LSU(2)$ by contractible open sets U_α. Consider the family of Hamiltonian systems $\left(T^*U_\alpha, \omega_{\mathrm{c}}^\alpha + \omega_{\mathrm{m}}^\alpha, H_{\mathrm{mb}}^\alpha\right)$, where $\omega_{\mathrm{c}}^\alpha + \omega_{\mathrm{m}}^\alpha$ denotes the restriction of $\omega_{\mathrm{c}} + \omega_{\mathrm{m}}$ to T^*U_α and H_{mb}^α is the restriction of the Hamiltonian function H_{mb}. The form ω_{m} is closed on $LSU(2)$, therefore its restriction to any contractible subset U_α is exact by the Poincaré lemma. We have $\omega_{\mathrm{m}}^\alpha = \mathrm{d}\theta^\alpha$.

Recall now the momentum shifting argument for the Hamiltonian systems with magnetic terms. Let M be a manifold and let $T_\theta: T^*M \to T^*M$ be a map defined by the formula $T_\theta(q, p_q) = (q, p_q - \theta_q)$. Let $H: T^*M \to \mathbb{R}$ be a Hamiltonian function and let $H_\theta(q, p_q) = H(q, p_q + \theta_q)$. Then T_θ pulls the function H_θ back to H and the canonical form ω_{c} back to the magnetically perturbed form $\omega_{\mathrm{c}} + \mathrm{d}\theta$. It is clear that a path $q(t): I \to T^*M$ is a solution of the Hamiltonian system $(T^*M, \omega_{\mathrm{c}} + \mathrm{d}\theta, H)$ if and only if it is also a solution of the Hamiltonian system $(T^*M, \omega_{\mathrm{c}}, H_\theta)$. Thus, for every $\alpha \in A$, the Hamiltonian system $\left(T^*U_\alpha, \omega_{\mathrm{c}}^\alpha + \omega_{\mathrm{m}}^\alpha, H_{\mathrm{mb}}^\alpha\right)$ is equivalent to the Hamiltonian system $(T^*U_\alpha, \omega_{\mathrm{c}}, H_\alpha)$, where $H_\alpha: T^*U \to \mathbb{R}$ is given by $H_\alpha(g, p_g) = (H_{\mathrm{mb}})_{/U_\alpha}\left(g, p_g + \theta_g^\alpha\right)$. By means of the Legendre transformation, we can now recast our restricted Hamiltonian systems into the Lagrangian form. We have the following result. A path $g(t): I \to U_\alpha$ is a solution of the

Hamiltonian system $(T^*U_\alpha, \omega_c, H_\alpha) \cong \left(T^*U_\alpha, \omega_c^\alpha + \omega_m^\alpha, H_{mb}^\alpha\right)$ if and only if it is an extremal of the Lagrangian functional $\mathcal{L}_\alpha : \{\text{paths on } U_\alpha\} \to \mathbb{R}$ given by

$$\mathcal{L}_\alpha(g(t)) = \int_I \left(\frac{1}{2}\|g_t g^{-1}\|^2 + \theta^\alpha(\dot{g}(t)) - \langle\langle \sigma, \mathrm{Ad}_{g(t)}(\tau)\rangle\rangle\right) \mathrm{d}t.$$

We can rewrite this Lagrangian somewhat more invariantly as

$$\mathcal{L}_\alpha(g(t)) = \int_I \left(\frac{1}{2}\|g_t g^{-1}\|^2 - \langle\langle \sigma, \mathrm{Ad}_{g(t)}(\tau)\rangle\rangle\right) \mathrm{d}t + \int_{g(I)} \theta^\alpha.$$

Note that θ^α is determined only up to a closed 1-form. But on the contractible U_α, every closed 1-form is also exact. For every 0-form (i.e. a function) β on U_α, we have

$$\int_{g(I)} \mathrm{d}\beta = \int_{\partial g(I)} \beta = \beta(g(b)) - \beta(g(a)).$$

Therefore, the Lagrangians $\mathcal{L}_\alpha$ corresponding to various possible choices of θ^α differ only by irrelevant constants when the endpoints of the paths $g(I)$ are fixed, and they do not differ at all when we consider the *closed* paths $g(S^1)$.

Now we will show that the family of local Lagrangians $\mathcal{L}_\alpha : \{\text{paths on } U_\alpha\} \to \mathbb{R}$ gives rise to a global Lagrangian

$$\mathcal{L} : \{\text{loops on } LSU(2)\} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1.$$

Let $g: S^1 \to LSU(2)$ be a loop in $LSU(2)$ and let $\hat{g}: D \to LSU(2)$ be an extension of g on the disc D, bounded by our S^1. Then $\hat{g}(D)$ is a two-dimensional submanifold in $LSU(2)$ whose boundary is the loop $g(S^1)$. Since $\hat{g}(D)$ is compact, it is covered by a finite subfamily $\{U_\alpha; \alpha \in A'\}$ of the covering $\{U_\alpha\}_{\alpha\in A}$. The disc D is two dimensional, therefore we can assume that at most three different U_α have non-empty intersection. Let $\bigcup_{\alpha\in A'} D_\alpha = D$ be a partition of the disc D into a union of curvilinear polygons D_α, such that for every $\alpha \in A'$ we have $\hat{g}(D_\alpha) \subset U_\alpha$ and such that the interiors of the polygons D_α are disjoint. A suitable partition $\bigcup_{\alpha\in A'} D_\alpha = D$ is given by the nerve of the covering $\{U_\alpha; \alpha \in A'\}$. In the group of one-dimensional chains in $LSU(2)$, we then have

$$g(S^1) = \partial\hat{g}(D) = \sum_{\alpha\in A'} (\partial\hat{g}(D_\alpha)).$$

For every $\alpha \in A'$, the theorem of Stokes gives $\int_{\partial\hat{g}(D_\alpha)} \theta^\alpha = \int_{\hat{g}(D_\alpha)} \omega_m$. But unlike θ^α, the form ω_m is globally defined. Therefore, we can define

$$\check{\mathcal{L}}(g(S^1)) = \int_{S^1} \left(\frac{1}{2}\|g_t g^{-1}\|^2 - \langle\langle \sigma, \mathrm{Ad}_{g(t)}(\tau)\rangle\rangle\right) \mathrm{d}t + \int_{\hat{g}(D)} \omega_m.$$

This functional is of course dependent on the choice of the extension $\hat{g}$ of the map $g: S^1 \to LSU(2)$. Let $\check{g}: D \to LSU(2)$ be another extension of g. Then the chain $\check{g}(D) - \hat{g}(D)$ is a smooth map

$$\check{g}(D) - \hat{g}(D) = \mathring{g}(S^2) : S^2 \longrightarrow LSU(2)$$

of a 2-sphere into $LSU(2)$. Now, $LSU(2)$ is diffeomorphic to $SU(2) \times \Omega SU(2)$, where $\Omega SU(2)$ denotes the group of the based loops in $SU(2)$. Since for the singular homology with integer coefficients we have $H_3(SU(2)) = H_3(S^3) = \mathbb{Z}$, we also get $H_2(\Omega SU(2) \cong H_3(SU(2)) = \mathbb{Z}$, and finally $H_2(LSU(2) = H_2(SU(2)) \times H_2(\Omega SU(2)) = \mathbb{Z}$.

The form ω_{m} is closed, but not exact. Therefore,

$$\int_{\check{g}(D_\alpha)} \omega_{\mathrm{m}} - \int_{\hat{g}(D_\alpha)} \omega_{\mathrm{m}} = \int_{\check{g}(D_\alpha)-\hat{g}(D_\alpha)} \omega_{\mathrm{m}} = \int_{\mathring{g}(S^2)} \omega_{\mathrm{m}} \in \mathbb{Z}$$

and, in general, this integer is different from zero. This shows that for different choices of the extension of the loop g on the disc D the values of the functional $\check{\mathcal{L}}$: {loops on $LSU(2)$} $\to \mathbb{R}$ can differ by integers. Therefore, the composition

$$\{\text{loops on } LSU(2)\} \xrightarrow{\check{\mathcal{L}}} \mathbb{R} \xrightarrow{\kappa} \mathbb{R}/\mathbb{Z} = S^1,$$

in which $\kappa : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is the natural projection, is independent of the choice of the map $\hat{g}$ extending the loop g. This proves that the Lagrangian functional $\mathcal{L} = \kappa \circ \check{\mathcal{L}}$ given by the formula

$$\mathcal{L}(g) = \int_{S^1} \left(\frac{1}{2} \| g_t g^{-1} \|^2 - \langle\langle \sigma, \mathrm{Ad}_g(\tau) \rangle\rangle \right) \mathrm{d}t + \int_{\hat{g}(D)} \omega_{\mathrm{m}}$$

is a well-defined single-valued map

$$\mathcal{L} : \{\text{loops on } LSU(2)\} \longrightarrow \mathbb{R}/\mathbb{Z},$$

as we have claimed in the statement of the theorem.

Finally, we have to show that the extremals of $\mathcal{L}$ are precisely the closed solutions of the Maxwell–Bloch Hamiltonian system $(T^*LSU(2), \omega_{\mathrm{c}} + \omega_{\mathrm{m}}, H_{\mathrm{cn}})$. But this is clear from our construction of $\mathcal{L}$. Inside every U_α, we have $\mathcal{L}_{/U_\alpha} = \mathcal{L}_\alpha$. Let $g(t)$ be an extremal of $\mathcal{L}$. Then its restriction to U_α is an extremal of $\mathcal{L}_\alpha$. We have shown that the corresponding path $(g(t), (g_t)^\flat)$ in the cotangent bundle T^*U_α is an integral path of the Hamiltonian vector field X_α defined by the Hamiltonian system $\left(T^*U_\alpha, \omega_{\mathrm{c}}^\alpha + \omega_{\mathrm{m}}^\alpha, H_{\mathrm{cn}}^\alpha\right)$. But, recalling that U_α is open in $LSU(2)$, we know the Hamiltonian vector field X_α coincides with the restriction of the Hamiltonian vector field X of our original Hamiltonian system $(T^*LSU(2), \omega_{\mathrm{c}} + \omega_{\mathrm{m}}, H_{\mathrm{cn}})$, which completes the proof of our theorem. □

Remark 3. The Lagrangian $\mathcal{L}$: {paths in $TLSU(2)$} $\to S^1$ is well defined only for *closed* paths, i.e. for temporally periodic solutions. For the Lagrangian description of the general non-periodic solutions, the extended configuration space $\widetilde{L}SU(2)$ must be used. The interested reader can compare our construction to the results in [30].

We shall conclude this paper with a comparison between the Maxwell–Bloch system and the Wess–Zumino–Witten–Novikov action. Let $X \subset \mathbb{R}^3$ be a closed two-dimensional orientable surface and let $f : X \to SU(2)$ be a smooth map. Denote by B the three-dimensional manifold bounded by the surface X; that is, $\partial B = X$. The Wess–Zumino–Witten–Novikov action is a two-dimensional conformal field theory given by the Lagrangian

$$\mathcal{L}_{\mathrm{wzwn}}(f) = \frac{1}{4\pi} \int_X (\nabla f) f^{-1} + \frac{1}{2\pi} \int_B \hat{f}^*(\Theta),$$

where $\hat{f} : B \to SU(2)$ is an extension of $f : X = \partial B \to SU(2)$ and $\Theta \in \Omega^3(SU(2))$ is the right-invariant 3-form whose value at the identity is given by

$$\Theta(\xi_1, \xi_2, \xi_3) = \langle \xi_1, [\xi_2, \xi_3] \rangle, \qquad \xi_1, \xi_2, \xi_3 \in T_e SU(2) = \mathfrak{su}(2).$$

In other words, the form Θ is the volume form on $SU(2) = S^3$ with respect to the natural round metric. One can immediately see that $\mathcal{L}_{\mathrm{wzwn}}$ is defined only up to addition of integers. Indeed, for two different choices $\hat{f}$ and $\check{f}$ of extensions, the chain $\check{f}(B) - \hat{f}(B)$ is a representative of a class in the homology group $H_3(SU(2)) = H_3(S^3) = \mathbb{Z}$. Since Θ is the volume form, it is closed, but not exact, and therefore $[\Theta]$ is a non-zero element in $H^3_{\mathrm{DR}}(SU(2))$. Thus, we have $\int_{(\check{f}-\hat{f})(B)} \Theta \in \mathbb{Z}$, as claimed.

Let now X be a sphere S^2 or a torus T^2. Both can be parametrized as closed paths of simple loops in obvious ways. We will denote the parameter of the closed path by $t \in S^1$ and the parameter on the simple loops by $x \in S^1$. The WZWN action for $f(t,x)\colon X \to SU(2)$ can then be written as

$$\mathcal{L}_{\mathrm{wzwn}}(f) = \frac{1}{4\pi}\int_X \left(\|f_t f^{-1}\|^2 + \|f_x f^{-1}\|^2\right) \mathrm{d}t\,\mathrm{d}x + \frac{1}{2\pi}\int_{\hat{f}(B)} \Theta.$$

We shall now compare the topologically non-trivial terms of the WZWN action and of the Maxwell–Bloch system. The relation between the forms $\Theta \in \Omega^3(SU(2))$ and $\omega_{\mathrm{m}} \in \Omega^2(LSU(2))$ is described by the following proposition. (See [30] for proof.)

Proposition 7. *Let* $\mathrm{ev}\colon S^1 \times LSU(2) \to SU(2)$ *be the evaluation map* $\mathrm{ev}(u, g(x)) = g(u)$ *and let* $\tau\colon \Omega^3(SU(2)) \to \Omega^2(LSU(2))$ *be defined by* $\tau(\alpha) = \int_{S^1} \mathrm{ev}^*(\alpha)$. *Then*

$$\omega_{\mathrm{m}} = \tau(2\pi\Theta) - \mathrm{d}\beta, \tag{44}$$

where β *is the 1-form on* $LSU(2)$ *given by*

$$\beta_g(X_g) = \frac{1}{4\pi}\int_0^{2\pi} \langle g_x g^{-1}, X_g g^{-1}\rangle\,\mathrm{d}x, \qquad X_g \in T_g LSU(2).$$

In particular, $[\tau(\Theta)] = [\omega_{\mathrm{m}}] \in H^2(LSU(2))$.

If we put formula (44) into expression (43) for the Lagrangian of the Maxwell–Bloch system, we get

$$\mathcal{L}(g) = \int_{S^1}\left(\frac{1}{2}\|g_t g^{-1}\|^2 - \langle\!\langle \sigma, \mathrm{Ad}_g(\tau(x))\rangle\!\rangle\right)\mathrm{d}t\,\mathrm{d}x + \int_{\hat{g}(D)} \tau(2\pi\Theta) - \int_{g(S^1)} \beta. \tag{45}$$

In the third term above, we have used Stokes' theorem and the fact that $\partial\hat{g}(D) = g(S^1)$. A loop $g\colon S^1 \to LSU(2)$ in the loop group $LSU(2)$ can be thought of as a map $f(t,x)\colon X \to SU(2)$, where X is a sphere or a torus. Formula (45), expressed in terms of the maps f rather than of the loops g, has the form

$$\mathcal{A}(f) = \int_X \left(\frac{1}{2}\|f_x f^{-1}\|^2 + \langle f_x f^{-1}, f_t f^{-1}\rangle - \langle \sigma, \mathrm{Ad}_f(\tau(x))\rangle\right)\mathrm{d}t\,\mathrm{d}x + \int_{\hat{f}(B)} \Theta, \tag{46}$$

in which the topologically non-trivial term is the same as in the WZWN action.

7. Conclusion

In this paper, a new Hamiltonian structure of the Maxwell–Bloch equations is constructed and some of its properties are studied. Our Hamiltonian structure stems from the representation of the Maxwell–Bloch equations as the equation of motion for a continuous chain of C Neumann oscillators parametrized by the single spatial variable x. The interaction among the oscillators is of magnetic type. This means that the acceleration of the oscillator on the location x_0 is influenced by the momenta rather than the positions of the neighbouring oscillators. Our Hamiltonian structure is of the form $(T^*LSU(2), \omega_{\mathrm{c}} + c\omega_{\mathrm{m}}, H_{\mathrm{mb}})$, where ω_{m} is the pull-back of the form $\widetilde{\omega}_{\mathrm{m}}$ on the loop group $LSU(2)$ via the natural projection $\pi\colon T^*LSU(2) \to LSU(2)$. The magnetic nature of the interaction among the oscillators is reflected in the perturbation $c\omega_{\mathrm{m}}$ of the canonical symplectic structure ω_{c}. The form $\widetilde{\omega}_{\mathrm{m}}$ is topologically non-trivial, but it is integral. It is in fact a generator of the cohomology group $H^2(LSU(2);\mathbb{Z}) \cong \mathbb{Z}$. By Weil's theorem, it is therefore the curvature of a connection on the topologically non-trivial principal $U(1)$-bundle $\widetilde{L}G \to LG$. The total space $\widetilde{L}SU(2)$ is precisely the central extension of the loop group $LSU(2)$. Therefore, the system $(T^*LSU(2), \omega_c + c\omega_m, H_{mb})$ is the symplectic quotient of the system $(T^*\widetilde{L}SU(2), \Omega_c, \widetilde{H})$, where Ω_c is the *canonical* symplectic form on $T^*\widetilde{L}SU(2)$ and $\widetilde{H}$ is the suitable Hamiltonian. The value of the moment map at which the quotient is taken is equal to c, that is, to the speed of light in the medium. In other words, the system $(T^*\widetilde{L}SU(2), \Omega_c, \widetilde{H})$ is the extension of $(T^*LSU(2), \omega_c + c\omega_m, H_{mb})$ in the sense of the Kaluza–Klein theory. The interaction force is geometrized on the $U(1)$-bundle $\widetilde{L}SU(2)$ over $LSU(2)$. This is reflected in the fact that the magnetically perturbed symplectic structure $\omega_c + c\omega_m$ on $LSU(2)$ lifts to the canonical structure on $\widetilde{L}SU(2)$. The conserved Kaluza–Klein charge in our case is the speed of light in the medium.

The Kaluza–Klein extension yields an easy way to find the Lagrangian for the Maxwell–Bloch equations. This Lagrangian is defined on the space of paths in the central extension $\widetilde{L}SU(2)$. We then construct the Lagrangian on the original configuration space $LSU(2)$. Here the non-trivial topology of the situation plays the crucial role. Namely, the Lagrangian contains the Wess–Zumino–Witten–Novikov term. Therefore, it is well defined only for temporally periodic solutions of the Maxwell–Bloch equations, while the Lagrangian on the Kaluza–Klein extension $\widetilde{L}SU(2)$ is well defined for arbitrary solutions.

We construct two families of solutions of the Maxwell–Bloch equations. One of these families nicely illustrates the relation between the Maxwell–Bloch and the C Neumann system. Our solutions are non-linear travelling waves whose constituent oscillator is the magnetic spherical pendulum in the same way as the harmonic oscillator is the constituent oscillator of the harmonic travelling waves. By the expression 'magnetic spherical pendulum' we call an electrically charged spherical pendulum moving in the field of a magnetic monopole situated at the centre of our sphere. The magnetic spherical pendulum is a symplectic quotient of a particular kind of circularly symmetric C Neumann system, the kind that figures in this paper. The well-known 2π-soliton occurs as a special case of our family of solutions. In this case, the constituent oscillator has to be reduced to the planar gravitational pendulum at the critical energy.

Our representation of Maxwell–Bloch equations as a chain of interacting oscillators and the associated Hamiltonian structure offer a starting point for many lines of further investigation. It is easily seen that this Hamiltonian system is invariant with respect to the natural action of the loop group $LU(1)$. More generally, Hamiltonian systems $(T^*LG, \omega_c + c\omega_m, H_{gmb})$ are invariant with respect to the actions of LH, where H are suitable subgroups of G. These actions yield various symplectic quotients. In a forthcoming paper, we intend to study some of these quotients and their properties. This topic is directly connected with the multilevel resonant light–matter interaction studied by Park and Shin in [16]. Another interesting topic is partial discretizations of the Maxwell–Bloch equations. If we discretize them with respect to the spatial variable, we get a discrete system of interacting C Neumann oscillators. In [32], we construct a large number of conserved quantities of such many-body systems. We intend to address different topics concerning the geometry and dynamics of such discretizations in future papers.

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